AFFINENESS AND RECONSTRUCTION IN HIGHER ZARISKI GEOMETRY

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Abstract. We explain how the geometric framework introduced in arXiv:2508.11621 [math.AG] provides a universal property for the 2-rings of perfect complexes on qcqs spectral or Dirac spectral schemes. As an application, given a qcqs spectral or Dirac spectral scheme X this produces a comparison morphism from $\operatorname{Spec}\operatorname{Perf}_X$ to X itself, which is moreover natural in X. When X is an ordinary qcqs scheme, this construction supplies a new proof of the Balmer-Thomason reconstruction of X from its space of thick subcategories, assuming the result for noetherian rings due to Neeman. As another application, we find spectral and Dirac spectral enhancements of support varieties arising for 2-rings in representation theory which "geometrize" the 2-rings that produce them. For example, given a finite group G over a field k, this produces a "spectral support variety" $\mathcal{V}_G$ such that $\operatorname{Perf}_{\mathcal{V}_G}$ maps into the stable module category of kG. We derive these results as a corollary of a general affineness criterion for 2-schemes which are covered by the Zariski spectra of rigid 2-rings: this states that such 2-schemes are affine if and only if they are quasicompact and quasiseparated.

Figure 1. *Creation of the Birds,* Remedios Varo, 1957[1]

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[1] https://www.wikiart.org/en/remedios-varo/creation-of-the-birds. Accessed on 10/17/2025.

Contents

1. Introduction

1.A. **Schemes in classical algebraic geometry.** Affine varieties, say over $\mathbb{C}$, are zero sets of collections of polynomial equations in $\mathbb{A}^n_{\mathbb{C}}$. Affine varieties have the decidedly nice feature that for two affine varieties V, W, regular maps between V and W may be identified with maps of commutative $\mathbb{C}$-algebras between $\mathcal{O}(W)$ and $\mathcal{O}(V)$, their corresponding rings of *regular functions* into the base field $\mathbb{C}$. Classical algebraic geometry over the complex numbers deals with the study of *complex varieties*, which are topological spaces admitting an open cover by affine varieties.

Given two complex varieties M and N, it is not necessarily the case that maps between M and N are determined by maps between their rings of regular functions into $\mathbb{C}$, although this *is* the case whenever the target is affine. Since they are locally affine, it is possible to compute maps between M and N by "gluing" maps between their corresponding affine open subsets. In this way the geometry of complex varieties is rendered accessible, but being a more general class they afford the construction of interesting objects to map to, or *moduli objects*, which bear import even if one is only interested in affine varieties.

Example (Complex projective space). The projective space $\mathbb{P}^{n-1}_{\mathbb{C}}$ is a complex variety whose points parametrize the linear subspaces of the vector space $\mathbb{C}^n$. Maps from a variety M into $\mathbb{P}^{n-1}_{\mathbb{C}}$ exactly parametrizes subspaces $E \subseteq \mathbb{A}^n_{\mathbb{C}} \times M$ which are fiberwise linear over M.

Following Zariski-Grothendieck, let us regard the objects above not merely as topological spaces, but in their following guise.

Recollection. The category of *locally ringed spaces* consists of pairs $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X satisfying a certain locality condition. Any ordinary commutative ring R arises as the global sections of a structure sheaf on a particular topological space $\operatorname{Spec} R$, its *classical Zariski spectrum*. $\operatorname{Spec} R$ along with its structure sheaf is the primary example of a locally ringed space, and the functor sending R to the locally ringed space $\operatorname{Spec} R$ participates in the following adjunction:

$$\Gamma : \{\text{Locally ringed spaces}\} \rightleftarrows \{\text{Commutative rings}\}^{\mathrm{op}} : \operatorname{Spec} \tag{1.1}$$

and the essential image of the functor Spec is said to comprise exactly the *affine schemes*.

Affine schemes implement affine varieties, which essentially appear within the Zariski spectra of their rings of regular functions. In this case, the adjunction of (1.1) exactly recovers the characteristic mapping property discussed in the first paragraph. It remains to implement the study of non-affine varieties in this language, for which one makes the following definition.

Recollection. Let $(X, \mathcal{O}_X)$ be a locally ringed space. If X admits an open cover $\{U_\alpha\}_{\alpha\in A}$ where each $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ may be identified with $\operatorname{Spec} R_\alpha$ for some commutative ring R_α, we say that $(X, \mathcal{O}_X)$ is a *scheme*.

Although the definitions above are motivated by complex algebraic geometry, the generality of the Zariski spectrum applies to the full gamut of interesting commutative rings which arise in arithmetic and geometry. As before, even if one is only concerned with the study of commutative rings, the category of schemes includes several interesting non-affine moduli to map affine objects to:

Example (Projective space). Given a commutative ring R, the following set

$$\frac{\{(\mathcal{L}, \phi) \mid \mathcal{L} \text{ an invertible module over } R,\ \phi : \bigoplus_{i=1}^{n} R \twoheadrightarrow \mathcal{L} \text{ an } R\text{-linear surjection}\}}{\{R - \text{linear isomorphisms which intertwine the given surjection}\}}$$

is naturally identified with the set of maps of locally ringed spaces from $\operatorname{Spec} R$ to the non-affine scheme $\mathbb{P}^{n-1}_{\mathbb{Z}}$.

To summarize, commutative rings appear as global sections of the structure sheaf on an affine scheme. In this form, geometric methods may be applied towards their study, and such methods will essentially employ non-affine objects.

1.B. **Affine moduli in higher Zariski geometry.** *Tensor-triangular geometry*, after [Bal11], is the study of tensor-triangulated categories (or *tt-categories*) via the use of algebro-geometric methods. This approach, at least philosophically, regards tt-categories as arising from the global sections of "structure sheaf of tt-categories" on the affine objects in a tt-analogue of algebraic geometry. This is unfortunately only a heuristic: there are problems with treating structure sheaves valued in tt-categories, as triangulated structure is poorly behaved with respect to forming limits in ordinary categories.

To pursue a geometric program which putatively includes non-affine objects, we must replace tensor-triangulated categories with a better behaved notion. To this end, we work in the ∞-category of 2-*rings*, denoted 2CAlg, defined to be the ∞-category of idempotent-complete stably symmetric monoidal ∞-categories with symmetric monoidal exact functors. In this setting, the idea that 2-rings appear as the affine objects of a categorified algebraic geometry can be made precise. This is accomplished in [HZG], utilising the formalism of *geometries* introduced in [DAGV].

Recollection. Let 2CAlg denote the ∞-category of 2-rings. There is an ∞-category of *locally 2-ringed spaces*, consisting of pairs X a topological space and $\mathcal{O} \in \operatorname{Shv}(X; 2\mathrm{CAlg})$ a sheaf of 2-rings on X, where the sheaf $\mathcal{O}$ is required to satisfy a certain locality condition[2]. There is a functor

$$\operatorname{Spec} : 2\mathrm{CAlg}^{\mathrm{op}} \to \{\text{Locally 2-ringed spaces}\}$$

known as the *Zariski spectrum*, which is the right adjoint in an adjunction of the following form

$$\Gamma : \{\text{Locally 2-ringed spaces}\} \rightleftarrows 2\mathrm{CAlg}^{\mathrm{op}} : \operatorname{Spec}$$

where Γ sends a pair $(X, \mathcal{O}_X)$ to the global sections of the sheaf $\mathcal{O}$ on X. The essential image of the functor Spec is said to comprise the *affine* 2-*schemes*.

If schemes in classical algebraic geometry provide interesting non-affine objects which apply towards the study of ordinary rings, it seems only natural that their obvious categorification ought have interesting applications to the study of 2-rings.

[2] see Subsection 2.C for a full discussion

Definition. A locally 2-ringed space $(X, \mathcal{O}_X)$ is a 2-*scheme* if it admits an open cover $\{U_\alpha\}$ such that the locally 2-ringed spaces $(U_\alpha, \mathcal{O}_X|_{U_\alpha}) \simeq \operatorname{Spec} \mathcal{K}_\alpha$ for $\mathcal{K}_\alpha \in 2\mathrm{CAlg}$. We say it is a *rigid* 2-*scheme* if the $\mathcal{K}_\alpha$ can be selected to be *rigid* 2-*rings*, see Definition 2.15.

The main result of this paper provides a characterization of rigid affine 2-schemes among the collection of all rigid 2-schemes. Recall that a topological space is *quasiseparated* if the intersection of any two quasicompact open subsets is quasicompact.

Theorem A. *Let $(X, \mathcal{O}_X)$ be a rigid 2-scheme. Then $(X, \mathcal{O}_X)$ is an affine 2-scheme if and only if the underlying topological space X is quasicompact and quasiseparated (or qcqs).*

The result above shows that higher Zariski geometry behaves in stark contrast to classical algebraic geometry, where most non-affine schemes of interest are qcqs, e.g., projective space over any base ring and any closed subscheme thereof. We view this as a feature rather than a bug: it demonstrates the category of 2-rings already contains the representing objects for many moduli problems. In our next result, we utilise our affineness criterion to demonstrate that a particular class of higher Zariski moduli problems arise as affine 2-schemes. In doing so, we provide a new universal property of the 2-ring of perfect complexes on a qcqs scheme.

Recollection. Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings, the higher-categorical enhancement of ordinary commutative rings. Recall that there is a limit-preserving functor

$$\mathcal{R}_{(-)} : 2\mathrm{CAlg} \to \mathrm{CAlg}$$

given by sending a 2-ring $\mathcal{K}$ to the endomorphism ring spectrum $\underline{\mathrm{End}}_{\mathcal{K}}(\mathbf{1})$ of the unit object $\mathbf{1} \in \mathcal{K}$. Thus, to any a locally 2-ringed space $(X, \mathcal{O})$, one may associate the pair $(X, \mathcal{R}_{\mathcal{O}})$, where $\mathcal{R}_{\mathcal{O}}$ is the sheaf of $\mathbb{E}_\infty$-rings given by

$$\mathcal{R}_{\mathcal{O}} : U \mapsto \mathcal{R}_{\mathcal{O}(U)} \coloneqq \underline{\mathrm{End}}_{\mathcal{O}(U)}(\mathbf{1}).$$

The space X along with this sheaf can be shown to be a *locally spectrally ringed space*: this is the ∞-category consisting of pairs $(X, \mathcal{O})$ of a topological space X equipped with a sheaf $\mathcal{O}$ of $\mathbb{E}_\infty$ rings on X satisfying a certain locality condition analogous to that of ordinary locally ringed spaces[3].

The previous recollection yields a restriction functor

$$\{\text{Locally 2-ringed spaces}\} \to \{\text{Locally spectrally ringed spaces}\}$$

given by sending a pair $(X, \mathcal{O})$ to the pair $(X, \mathcal{R}_{\mathcal{O}})$. This restriction functor will give rise to a natural family of moduli problems:

Definition. Let X be a spectral scheme[4] (of which ordinary schemes are an example). Define the *relative spectrum* of X to be the unique locally 2-ringed space $\operatorname{Spec}^2_1 X$ satisfying the following:

$$\operatorname{map}((Y, \mathcal{O}_Y), \operatorname{Spec}^2_1 X) \simeq \operatorname{map}((Y, \mathcal{R}_{\mathcal{O}_Y}), X)$$

for $(Y, \mathcal{O}_Y)$ any arbitrary locally 2-ringed space. In the above, the left hand mapping space is taken in locally 2-ringed spaces, and the right-hand mapping space is taken in locally spectrally ringed spaces.

[3] see Subsection 2.A for a full discussion

[4] see Definition 3.15 for the full definition.

The restriction morphism from locally 2-ringed spaces to locally spectrally ringed spaces may also be obtained via the general framework of *geometries* outlined in [DAGV]. In Section 2.1 of *loc. cit.* the author constructs the relative spectrum of *any* locally spectrally ringed space, and functorially so, as a left adjoint to the restriction functor (with the caveat that one needs to work with locally 2/spectrally ringed *topoi* in lieu of topological spaces). These results in fully generality are recalled in Subsection 2.C.

The upshot is that relative spectra always exist, and our next result show that these moduli problems are in fact associated to affine schemes.

Theorem B. *Given a qcqs spectral scheme X, its relative spectrum $\mathrm{Spec}_1^2 X$ is a rigid qcqs 2-scheme. Hence, it is an affine 2-scheme by Theorem A. The global sections of $\mathrm{Spec}_1^2 X$ may be naturally identified with the 2-ring Perf_X of perfect complexes on X (Definition 5.1), furnishing an identification*

$$\mathrm{Spec}_1^2 X \simeq \mathrm{Spec}\,\mathrm{Perf}_X\,.$$

These equivalences are natural in the spectral scheme X.

We also show that the result above holds in the case of *Dirac* spectral schemes, see Definition 3.15.

Remark 1.2. Theorem B can be regarded as an instance of the "1-affineness" philosophy of [Gai14] for qcqs spectral schemes. In fact, alongside the the descent results of [HZG, §5], Theorem B will recover the fact that such spectral schemes are 1-affine, demonstrated in [Gai14, §5]. It will also supply the expected extension of these results to the Dirac spectral case.

In [HZG], it is shown that the functor Spec from 2-rings to locally 2-ringed spaces restricts to a fully faithful functor on the full subcategory $2\mathrm{CAlg}_{\mathrm{rig}} \subseteq 2\mathrm{CAlg}$ of rigid 2-rings. As an immediate corollary, we obtain the following, which is a new universal property for the 2-ring of perfect complexes on a spectral scheme.

Corollary C. *For a rigid 2-ring $\mathcal{K}$ and a qcqs spectral scheme X, there is an identification*

$$\mathrm{map}_{2\mathrm{CAlg}}(\mathrm{Perf}_X, \mathcal{K}) \simeq \mathrm{map}((\mathrm{Spec}\,\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\mathrm{Spec}\,\mathcal{K}}}), X)$$

where the right-hand mapping space is taken in locally spectrally ringed spaces. Moreover, these equivalences are natural in both the 2-ring $\mathcal{K}$ and the spectral scheme $\mathcal{X}$.

As above, we also demonstrate that Corollary C holds in the Dirac spectral setting. In the next subsection, we discuss some immediate consequences of this result.

1.C. **Reconstruction of schemes and geometrization of 2-rings.** Given a qcqs ordinary scheme X, a pioneering result of tensor-triangular geometry due to Thomason [Tho97], building off work of Hopkins [Hop87] and Neeman [NB92], identifies the underlying space of $\mathrm{Spec}\,\mathrm{Perf}_X$ with the underlying space of X.

In [Bal02] Balmer provides a way to recover X as a locally ringed space, by working with a tt-categorical implementation of the structure sheaf on $\mathrm{Spec}\,\mathrm{Perf}_X$. Using the framework above, we are able to recover and enhance his result to include derived objects.

Observation. Let X be a spectral scheme. Then recall that the universal property of Corollary C furnishes a comparison map of locally spectrally ringed spaces

$$\gamma_X : (\mathrm{Spec}\,\mathrm{Perf}_X, \mathcal{R}_{\mathcal{O}_{\mathrm{Spec}\,\mathrm{Perf}_X}}) \to X$$

induced by the identity functor on Perf_X, which is moreover natural in the spectral scheme X.

Theorem D. *If X is an ordinary qcqs scheme regarded as a spectral scheme, the comparison map γ_X constructed above is an equivalence.*

We remark that this also provides a new proof of Thomason's result, assuming Neeman's classification result for the case where X is an ordinary affine noetherian scheme. We touch upon the qualitative differences to Thomason's approach in Remark 5.14. Finally, we show that Corollary C gives a simple method for "geometrizing" certain 2-rings.

Theorem E. *Let $\mathcal{K}$ be a 2-ring such that the locally spectrally ringed space $(\operatorname{Spec}\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\operatorname{Spec}\mathcal{K}}})$ is itself a spectral scheme. Then there is a fully faithful embedding*

$$\operatorname{Perf}_{(\operatorname{Spec}\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\operatorname{Spec}\mathcal{K}}})} \hookrightarrow \mathcal{K}$$

induced by the identity functor on $(\operatorname{Spec}\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\operatorname{Spec}\mathcal{K}}})$ via the universal property of Corollary C.

Theorem E is a special case of a slightly stronger statement, which is collected in Theorem 6.9. However, just this case is already sufficient for a broad swath of applications:

Example. Let G be a finite group and k a field of characteristic dividing the order of G. The *stable module category* of kG, denoted St_{kG}, is a 2-ring which exactly captures the failure of the representation theory of G over k to be semisimple (see Example 6.7 for a precise definition). In the linked example, we show that $(\operatorname{Spec}\operatorname{St}_{kG}, \mathcal{R}_{\mathcal{O}})$ is itself a spectral scheme whose underlying classical scheme is identified with $\operatorname{Proj} H^*(G,k)$. We refer to this spectral scheme as the *spectral support variety*, and denote it by $\mathcal{V}_G$. Theorem E now supplies a fully faithful embedding

$$\operatorname{Perf}_{\mathcal{V}_G} \hookrightarrow \operatorname{St}_{kG}$$

which is moreover functorial in the group G.

Such a map is constructed in [Mat16, §9] by hand for the case of G elementary abelian, and in fact Mathew's observation in *loc. cit.* formed the genesis for this paper's investigation. In [Mat15b], Mathew utilizes this embedding to provide a new proof of Dade's theorem via spectral algebraic geometry. A shadow of the same is recovered by work of [Bal10a], which constructs an injective homomorphism from the Picard group of lines bundles on the variety $\operatorname{Proj} H^*(G,k)$ to the group of invertible objects of St_{kG}, after inverting the characteristic of the base field, and also by hand. The embedding $\operatorname{Perf}_{\mathcal{V}_G} \hookrightarrow \operatorname{St}_{kG}$ unifies these two observations, and moreover supplies a functoriality result for Balmer's homomorphism (which, per the author's knowledge, was hitherto unclear). We will expand upon these considerations in [Che], where we also explore applications to computing the group of *torsion-free endotrivial modules* of kG.

Finally, we note that all of the results above work in the Dirac spectral setting. We have chosen this level of generality owing to recent work of Balmer-Gallauer [BG25] which shows that the Zariski spectrum of the *derived category of permutation modules* of G over k is a Dirac spectral scheme for G an elementary abelian group. We expound upon this example in Example 6.8.

1.D. **Overview.** Section 2 contains the necessary recollections on geometries following [DAGV], in addition to results on the Zariski geometry of 2-rings following [HZG]. In Section 3 we collect some preliminaries on relative spectra from [DAGV] which will enable their computation via descent. Our key result here is Proposition 3.17, which demonstrates that the relative spectra of schemes are 0-localic. Our first main theorem, Theorem 4.7, is proved in Section 4. Section 5 is dedicated to the proofs of Theorem B and Theorem D. Finally, Corollary C and Theorem E are recorded in Section 6.

1.E. **Conventions.** We follow the notational conventions of [HZG]. *This color modifier* is used to mark instances of important definitions or notations for the reader's convenience. When writing an adjunction between ∞-categories as

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R,$$

it is always understood that L is left adjoint to R, in symbols $L \dashv R$, unless otherwise specified.

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2. Recollections on the Geometry of Rings and 2-Rings

2.A. **Geometries.** In this subsection we recall the classical Zariski and Dirac geometries on CAlg as constructed in [DAGVII] and [HZG, §3]. We then go on to recall the construction of the absolute spectrum and the relative spectrum associated to a morphism of geometries.

Definition 2.1. Let $\mathcal{G}$ be a small ∞-category, $\mathcal{G}^{\mathrm{ad}}$ a wide subcategory of $\mathcal{G}$, and τ a Grothendieck topology on $\mathcal{G}$. We say that $(\mathcal{G}^{\mathrm{ad}}, \tau)$ is an *admissibility structure* on $\mathcal{G}$ or that $(\mathcal{G}, \mathcal{G}^{\mathrm{ad}}, \tau)$ is a *geometry* if the conditions below are satisfied.

(1) $\mathcal{G}$ has finite limits and is idempotent complete.
(2) τ is generated by morphisms in $\mathcal{G}^{\mathrm{ad}}$.
(3) $\mathcal{G}^{\mathrm{ad}}$ is closed under base changes in $\mathcal{G}$.
(4) If f is a retract of g in $\mathcal{G}^{[1]}$ satisfying $g \in \mathcal{G}^{\mathrm{ad}}$, then $f \in \mathcal{G}^{\mathrm{ad}}$ as well.

We refer to morphisms in $\mathcal{G}^{\mathrm{ad}}$ and covers in τ as admissible morphisms and admissible covers, respectively.

Definition 2.2. The *classical Zariski geometry* on (nonconnective) $\mathbb{E}_\infty$ ring spectra consists of the following data:

(1) $\mathcal{G}_{\mathrm{cZar}} = \mathrm{CAlg}^{\omega,\mathrm{op}}$, the opposite of the ∞-category of compact $\mathbb{E}_\infty$ rings.
(2) Admissible morphisms correspond to localization maps $R \to R[x^{-1}]$ for $x \in \pi_0 R$
(3) A finite collection $\{R \to R[x_i^{-1}]\}_{i \in I}$ generates a covering sieve if the set $\{x_i\}_{i \in I} \subset \pi_0 R$ generates the unit ideal.

Remark 2.3. There is a variant of the above definition, $\mathcal{G}^{\mathrm{cn}}_{\mathrm{cZar}}$, which is restriction of the admissibility structure above to the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$ ring spectra. Note that the inclusion $\mathrm{CAlg}^{\mathrm{cn}} \hookrightarrow \mathrm{CAlg}$ restricts to compact objects, and as such induces a morphism of geometries $\mathcal{G}^{\mathrm{cn}}_{\mathrm{cZar}} \to \mathcal{G}_{\mathrm{cZar}}$. We will not consider this variant here.

Definition 2.4. The *Dirac geometry* on $\mathbb{E}_\infty$-ring spectra consists of the following data:

(1) $\mathcal{G}_{\mathrm{Dir}} = \mathrm{CAlg}^{\omega,\mathrm{op}}$, the opposite of the ∞-category of compact $\mathbb{E}_\infty$ rings.
(2) Admissible morphisms correspond to localization maps $R \to R[x^{-1}]$ for homogenous elements $x \in \pi_* R$

(3) A finite collection $\{R \to R[x_i^{-1}]\}_{i\in I}$ generates a covering sieve if the set $\{x_i\}_{i\in I} \subset \pi_* R$ generates the unit ideal.

Recall that any geometry $\mathcal{G}$ has an associated ∞-category of $\mathcal{G}$-structured ∞-topoi.

Definition 2.5. Given a geometry $\mathcal{G}$, we write $\mathrm{LTop}(\mathcal{G})$ to denote the ∞-category of *$\mathcal{G}$-structured ∞-topoi*. This is the ∞-category consisting of pairs $(\mathcal{X}, \mathcal{O})$ where $\mathcal{X} \in \mathrm{LTop}$ is an ∞-topos and $\mathcal{O} \in \mathrm{Shv}(\mathcal{X}; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}))$ is an $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$-valued sheaf satisfying a particular locality condition with respect to the admissibility structure on $\mathcal{G}$. A morphism $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a pair

$$f^* : \mathcal{X} \to \mathcal{Y} \in \mathrm{LTop}^{[1]}, \ \phi : f^*\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Y} \in \mathrm{Shv}(\mathcal{Y}; \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$$

where f^* is used both to refer to left-exact left adjoint in $\mathrm{LTop}^{[1]}$ and its induced pullback functor on $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$-valued sheaves, and ϕ is itself required to satisfy a particular locality condition with respect to the admissibility structure on $\mathcal{G}$.

We have opted to omit specifics in the above, and refer to [DAGV, §1.2] or [HZG, §3] for a full recollection of these definitions. Rather than do this ourselves, let us quickly recall the behaviour of the ∞-categories of structured topoi obtained from the geometries above.

Example 2.6. The ∞-category $\mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})$ of consists of pairs $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ where $\mathcal{X} \in \mathrm{LTop}$ and $\mathcal{O}_\mathcal{X} \in \mathrm{Shv}(\mathcal{X}; \mathrm{CAlg})$ a sheaf of rings which is *local* in the following senses:

(1) The sheaf $\mathcal{O}_\mathcal{X}$ is locally nontrivial, i.e., it is not identically the 0 ring.
(2) Let $\mathcal{O}_\mathcal{X}^\times$ denote the sheaf of units of $\mathcal{O}_\mathcal{X}$, and let $e : \mathcal{O}_\mathcal{X}^\times \to \mathcal{O}_\mathcal{X}$ denote the canonical inclusion. Then $e \coprod (1-e) : \mathcal{O}_\mathcal{X}^\times \coprod \mathcal{O}_\mathcal{X}^\times \to \mathcal{O}_\mathcal{X}$ is an effective epimorphism.

A morphism $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ consists of a left-exact left adjoint $f^* : \mathcal{X} \to \mathcal{Y}$ along with a morphism $f^*\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Y} \in \mathrm{Shv}(\mathcal{O}_\mathcal{Y}; \mathrm{CAlg})$ such that the following square is Cartesian:

$$\begin{array}{ccc} f^*\mathcal{O}_\mathcal{X}^\times & \longrightarrow & \mathcal{O}_\mathcal{Y}^\times \\ \downarrow & & \downarrow \\ f^*\mathcal{O}_\mathcal{X} & \longrightarrow & \mathcal{O}_\mathcal{Y}. \end{array}$$

Remark 2.7. Given a point $x^* : \mathcal{X} \to \mathcal{S}$, it is easy to check that the stalk $\mathcal{O}_{\mathcal{X},x} \in \mathrm{CAlg}$ is a *local $\mathbb{E}_\infty$-ring* in the sense that it is a local ring on π_0, and that any morphism in $\mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})$ induces a π_0-local morphism of $\mathbb{E}_\infty$ rings on stalks.

Example 2.8. The ∞-category $\mathrm{LTop}(\mathcal{G}_{\mathrm{Dir}})$ consists of pairs $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ where $\mathcal{X} \in \mathrm{LTop}$ and $\mathcal{O}_\mathcal{X} \in \mathrm{Shv}(\mathcal{X}; \mathrm{CAlg})$ a sheaf of rings satisfying the conditions of [DAGV, Definition 1.2.8] for the geometry $\mathcal{G}_{\mathrm{Dir}}$. A morphism $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ consists of a left-exact left adjoint $f^* : \mathcal{X} \to \mathcal{Y}$ along with a morphism $f^*\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Y} \in \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{O}_\mathcal{Y})$ which is a local transformation of $\mathcal{G}_{\mathrm{Dir}}$-structures in the sense of *loc. cit.*

Remark 2.9. Given any point $x^* : \mathcal{X} \to \mathcal{S}$ the stalk $\mathcal{O}_{\mathcal{X},x} \in \mathrm{CAlg}$ is a *Dirac-local $\mathbb{E}_\infty$-ring* in the sense that it is a Dirac-local ring on π_*, and any morphism in $\mathrm{LTop}(\mathcal{G}_{\mathrm{Dir}})$ induces a π_* Dirac-local morphism of $\mathbb{E}_\infty$ rings on stalks.

Recall that a transformation of geometries $\mathcal{G} \to \mathcal{G}'$ is, informally, a left-exact functor which respects admissible morphisms and admissible covers. The following functor is constructed in [DAGV, §2.1].

Definition 2.10. Given a transformation of geometries $\mathcal{G} \to \mathcal{G}'$, we write $\mathrm{Spec}_{\mathcal{G}}^{\mathcal{G}'} : \mathrm{LTop}(\mathcal{G}) \to \mathrm{LTop}(\mathcal{G}')$ to denote the left adjoint to the natural forgetful functor $\mathrm{LTop}(\mathcal{G}') \to \mathrm{LTop}(\mathcal{G})$. We will call this the *relative spectrum* functor.

Given a geometry $\mathcal{G}$, one always has access to $\mathcal{G}^{\mathrm{disc}}$ the *discrete geometry* on $\mathcal{G}$, given by the geometry with only equivalences as admissible covers. There is always a transformation of geometries $\mathcal{G}^{\mathrm{disc}} \to \mathcal{G}$, which informally gives rise to a sequence of functors

$$\mathrm{LTop}(\mathcal{G}) \xrightarrow{\mathrm{fgt}} \mathrm{LTop}(\mathcal{G}^{\mathrm{disc}}) \xrightarrow{\Gamma(-,\mathcal{O})} \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}).$$

The latter functor $\Gamma(-,\mathcal{O}) : \mathrm{LTop}(\mathcal{G}^{\mathrm{disc}}) \to \mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ admits a left adjoint, sending $R \in \mathrm{Ind}(\mathcal{G})$ to the pair $(\mathcal{S}, \underline{R}) \in \mathrm{LTop}(\mathcal{G}^{\mathrm{disc}})$.

Definition 2.11. We write $\mathrm{Spec}^{\mathcal{G}} : \mathrm{Ind}(\mathcal{G}) \to \mathrm{LTop}(\mathcal{G})$ to denote the composite

$$\mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \xrightarrow{(\mathcal{S},\underline{R})} \mathrm{LTop}(\mathcal{G}^{\mathrm{disc}}) \xrightarrow{\mathrm{Spec}^{\mathcal{G}}_{\mathcal{G}^{\mathrm{disc}}}} \mathrm{LTop}(\mathcal{G})$$

which we refer to as the *absolute spectrum* functor.

The following follows from the existence of the adjoints above.

Theorem 2.12. *One has an adjunction*

$$\mathrm{Spec}^{\mathcal{G}} : \mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \rightleftarrows \mathrm{LTop}(\mathcal{G}) : \Gamma_{\mathcal{G}}$$

where $\Gamma_{\mathcal{G}} \coloneqq \Gamma(-,\mathcal{O})$. *Namely, there is a natural equivalence*

$$\mathrm{map}_{\mathrm{LTop}(\mathcal{G})}(\mathrm{Spec}^{\mathcal{G}}(-), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \simeq \mathrm{map}_{\mathcal{G}}(-, \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$$

of functors from $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}})$ *to* $\widehat{\mathcal{S}}$.

Example 2.13. For $R \in \mathrm{CAlg}$, one has an identification $\mathrm{Spec}^{\mathcal{G}_{\mathrm{cZar}}} \simeq (\mathrm{Shv}(\mathrm{Spec}\,\pi_0 R), \mathcal{O}_{\mathrm{Spec}\,R})$ of locally spectrally ringed topoi. In particular, the identification $\Gamma(\mathrm{Spec}\,\pi_0 R, \mathcal{O}_{\mathrm{Spec}\,R}) \simeq R$ implies that the absolute spectrum functor is fully faithful. This latter statement implies the subcanonicity of the Zariski topology on CAlg.

2.B. **2-rings.** Recall the tensor product on $\mathrm{Cat}^{\mathrm{perf}}$ of [BFN10, §4.1.2], informally characterized by the fully faithful inclusion $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n \to \mathcal{C}') \hookrightarrow \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{C}')$ with essential image given by functors which are exact in each variable.

Definition 2.14. We let 2CAlg denote the underlying ∞-category associated to $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$. The objects of 2CAlg are referred to as *2-rings*.

Concretely, a 2-ring is a small symmetric monoidal idempotent-complete stable ∞-category $\mathcal{K} = (\mathcal{K}, \otimes, \mathbf{1})$ such that $\otimes$ is exact in both variables, and a morphism between 2-rings $\mathcal{K}$ and $\mathcal{L}$ is a symmetric monoidal exact functor $f : \mathcal{K} \to \mathcal{L}$. The homotopy category of a 2-ring naturally has the structure of a small idempotent-complete tt-category.

Definition 2.15. The full subcategory $\mathrm{2CAlg}_{\mathrm{rig}} \subseteq \mathrm{2CAlg}$ of *rigid 2-rings* comprises of those 2-rings $\mathcal{K}$ such that any object $x \in \mathcal{K}$ admits a dual $x^{\vee}$.

Example 2.16. Given an $\mathbb{E}_\infty$ ring spectrum R, the 2-ring of perfect complexes Perf_R is a rigid 2-ring. More generally, the subcategory $\mathrm{2CAlg}_{\mathrm{rig}} \subseteq \mathrm{2CAlg}$ is coreflective and hence closed under limits, see [HZG, Proposition 2.37].

Definition 2.17. Given a 2-ring $\mathcal{K}$, a *thick tensor ideal* or *tt-ideal* of $\mathcal{K}$ is a stable subcategory $\mathcal{I} \subseteq \mathcal{K}$ which is closed under retracts, and is moreover closed under tensoring with any object of $\mathcal{K}$. Given a subset $\mathcal{G} \subseteq \mathcal{K}$, we write $\langle \mathcal{G} \rangle$ to denote the smallest thick tt-ideal containing $\mathcal{G}$.

Definition 2.18. Given a 2-ring $\mathcal{K}$ and a thick tensor-ideal $\mathcal{I} \subseteq \mathcal{K}$, the *Karoubi quotient* of $\mathcal{K}$ is the initial object of the full subcategory of $2\mathrm{CAlg}_{\mathcal{K}/}$ consisting of the symmetric monoidal functors which send every object of $\mathcal{I}$ to 0. We denote this object by $\mathcal{K}/\mathcal{I}$.

Remark 2.19. Karoubi quotients always exist for any tt-ideal, and are the idempotent-complete incarnations of Verdier localizations. Outside of the monoidal setting, these are treated in [Cal+25, Appendix A.3]. For a quick overview of the basic theory in 2CAlg, we refer the reader to [HZG, §2].

2.C. **Prerequisite results on the Zariski geometry of 2-rings.** We will need to utilise the following facts, all of which are imported from [HZG, §3-4].

Definition 2.20 ([HZG, Theorem A])**.** The following data defines a geometry, known as the *Zariski geometry*, on commutative 2-rings.

(1) $\mathcal{G}_{\mathrm{Zar}} \coloneqq 2\mathrm{CAlg}^{\omega,\mathrm{op}}$ is the opposite of the ∞-category of compact 2-rings.
(2) A morphism $\mathcal{K} \to \mathcal{K}'$ in $(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{op}}$ is called admissible if it corresponds to a Karoubi quotient $\mathcal{K} \to \mathcal{K}/\mathcal{I}$.
(3) A finite collection of admissible morphisms $\{f_i : \mathcal{K} \to \mathcal{K}_i\}_{i\in I}$ is declared to generate a covering sieve if for every $x \in \bigcap_{i\in I} \ker f_i$, there exists an n so that $x^{\otimes n} = 0$.

Example 2.21. The ∞-category $\mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$ of consists of pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X} \in \mathrm{LTop}$ and $\mathcal{O}_{\mathcal{X}} \in \mathrm{Shv}(\mathcal{X}; 2\mathrm{CAlg})$ is a sheaf of 2-rings satisfying the conditions of [DAGV, Definition 1.2.8] for the geometry $\mathcal{G}_{\mathrm{Zar}}$. A morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is as above.

Remark 2.22. Given any point $x : \mathcal{X} \to \mathcal{S}$ the stalk $\mathcal{O}_{\mathcal{X},x} \in 2\mathrm{CAlg}$ is a *local 2-ring* in the sense that if $x \otimes y \in \mathcal{O}_{\mathcal{X},x}$ is tensor-nilpotent, then either x or y is tensor-nilpotent. Any morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$ induces nil-conservative morphisms on stalks, namely morphisms with kernels consisting of tensor-nilpotent elements.

As suggested by the terminology, one may compare the classical Zariski geometry of $\mathbb{E}_\infty$ ring spectra and the Zariski geometry of 2-rings.

Proposition 2.23 ([HZG, Proposition 4.33])**.** *The assignment $A \mapsto \mathrm{Perf}_A$ yields morphisms of geometries $\mathcal{G}_{\mathrm{cZar}} \to \mathcal{G}_{\mathrm{Zar}}$ and $\mathcal{G}_{\mathrm{Dir}} \to \mathcal{G}_{\mathrm{Zar}}$.*

From the proposition above, one obtains a series of morphisms $\mathcal{G}_{\mathrm{cZar}} \subseteq \mathcal{G}_{\mathrm{Dir}} \to \mathcal{G}_{\mathrm{Zar}}$. The associated restriction functors from $\mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$ to $\mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})$ (or $\mathrm{LTop}(\mathcal{G}_{\mathrm{Dir}})$) can be identified through the helpful proposition below.

Definition 2.24. We write $\mathcal{R}_{(-)}$ to indicate the functor sending $\mathcal{K} \in 2\mathrm{CAlg}$ to the endomorphism ring spectrum $\hom_{\mathcal{K}}(\mathbf{1}, \mathbf{1}) \in \mathrm{CAlg}$. This functor participates in an adjunction of the form

$$\mathrm{Perf} : \mathrm{CAlg} \rightleftarrows 2\mathrm{CAlg} : \mathcal{R}_{(-)}$$

where the left adjoint is fully faithful, see for example [HZG, Construction 4.29].

Proposition 2.25 ([HZG, Lemma 3.22])**.** *Given a $\mathcal{G}_{\mathrm{Zar}}$-structured ∞-topos $(\mathcal{X}, \mathcal{O}) \in \mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$, its associated restriction to $\mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})$ (or $\mathrm{LTop}(\mathcal{G}_{\mathrm{Dir}})$) may be identified with the pair $(\mathcal{X}, \mathcal{R}_{\mathcal{O}})$ where $\mathcal{R}_{\mathcal{O}}$ is the composite*

$$\mathcal{R}_{(-)} \circ \mathcal{O} : \mathcal{X}^{\mathrm{op}} \to \mathrm{CAlg}$$

In [HZG] it is shown that the Zariski geometry of a 2-ring is captured by its *Balmer spectrum*. We recall this notion and the essential features required below. For a more detailed recollection, we refer either to [HZG, Section 4.1] or to Balmer's original paper [Bal05].

Recollection 2.26. Let $\mathcal{K}$ be a 2-ring. The *Balmer spectrum* of $\mathcal{K}$ is the topological space $\operatorname{Spc}\mathcal{K}$ whose underlying set is given by

$$\{\mathcal{P} \mid \mathcal{P} \subseteq \mathcal{K} \text{ is a prime tt-ideal}\}$$

Where by *prime tt-ideal* we mean that $\mathcal{P}$ is a tt-ideal and $\forall x, y \in \mathcal{K},\ x \otimes y \in \mathcal{P} \implies x \in \mathcal{P}$ or $y \in \mathcal{P}$. The topology of $\operatorname{Spc}\mathcal{K}$ is generated by a basis of open subsets $U(a) \coloneqq \{\mathcal{P} \in \operatorname{Spc}\mathcal{K} \mid a \in \mathcal{P}\}$. It can be shown that the basic open subsets are exactly the quasicompact open subsets, and with this topology the Balmer spectrum is a quasicompact, quasiseparated spectral space.

The following theorem is a combination of [HZG, Theorem C, Theorem D].

Theorem 2.27. *Let* $\mathcal{K} \in \mathrm{2CAlg}$.

(1) There is a natural identification of underlying topoi $\operatorname{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\mathcal{K} \simeq \operatorname{Shv}(\operatorname{Spc}\mathcal{K})$*, where the* $\operatorname{Spc}\mathcal{K}$ *refers to the Balmer spectrum of the tt-category* $\operatorname{ho}\mathcal{K}$.

(2) If $\mathcal{K}$ *is moreover assumed to be rigid, then the associated structure sheaf on* $\operatorname{Spec}\mathcal{K}$ *may be identified with the unique* $\mathrm{2CAlg}$*-valued sheaf on* $\operatorname{Spc}\mathcal{K}$ *which sends quasicompact opens of the form* $U(a) \subseteq \operatorname{Spc}\mathcal{K},\ a \in \mathcal{K}$ *to the Karoubi quotients* $\mathcal{K}/\langle a\rangle$.

Note that the global sections of the structure sheaf on $\operatorname{Spec}\mathcal{K}$ are identified with sections on the open subset $U(0) \subseteq \operatorname{Spc}\mathcal{K}$. From this, we immediately obtain the following.

Corollary 2.28. *Let* $\mathcal{K} \in \mathrm{2CAlg}_{\mathrm{rig}}$ *a rigid 2-ring. Then the counit of the adjunction* $\operatorname{Spec} \dashv \Gamma$ *yields an equivalence* $\mathcal{K} \simeq \Gamma(\operatorname{Spec}\mathcal{K}, \mathcal{O}_{\operatorname{Spec}\mathcal{K}})$*. In particular, the functor* $\operatorname{Spec} : \mathrm{2CAlg}_{\mathrm{rig}} \to \mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$ *is fully faithful.*

Notation 2.29. We will henceforth write $\operatorname{Spec}\mathcal{K} \coloneqq \operatorname{Spec}^{\mathcal{G}_{\mathrm{Zar}}}\mathcal{K}$ for $\mathcal{K} \in \mathrm{2CAlg}$, and will refer to the underlying space by $|\operatorname{Spec}\mathcal{K}|$. Its associated structure sheaf will be denoted $\mathcal{O}_{\mathcal{K}}$.

3. Preliminaries on relative spectra and descent

In this section we collect certain results of [DAGV] which will allow us to compute relative spectra by descent. We use this to write down the key observation that the relative spectra of 0-localic ∞-topos arising from spectral schemes (Definition 3.15) are themselves 0-localic.

3.A. Étale maps.

Recall the following definition of [HTT, §6.3.5].

Definition 3.1. A map $f_* : \mathcal{X} \to \mathcal{Y} \in \mathrm{LTop}^{[1]}$ is said to be *étale* if it admits a factorization

$$\mathcal{X} \xrightarrow{\pi^*} \mathcal{X}_{/U} \xrightarrow{\sim} \mathcal{Y}$$

where π^* is right adjoint to the projection $\pi_! : \mathcal{X}_{/U} \to \mathcal{X}$ for some $U \in \mathcal{X}$.

Note that under these conditions, π^* is itself in $\mathrm{LTop}^{[1]}$.

Definition 3.2. Let $\mathcal{G}$ be a geometry. A morphism $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in \mathrm{LTop}(\mathcal{G})^{[1]}$ is said to be *étale* if the following conditions are satisfied:

(1) The underlying geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$ is étale.
(2) The induced morphism $f^*\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$ is an equivalence in $\operatorname{Shv}(\mathcal{Y}; \operatorname{Ind}(\mathcal{G}^{\mathrm{op}}))$.

Notation 3.3. We write $\mathrm{LTop}_{\text{ét}} \subset \mathrm{LTop}$ and $\mathrm{LTop}(\mathcal{G})_{\text{ét}} \subset \mathrm{LTop}(\mathcal{G})$ to indicate the wide subcategories spanned by the étale morphisms. Note that the induced forgetful functor $\mathrm{LTop}(\mathcal{G})_{\text{ét}} \to \mathrm{LTop}_{\text{ét}}$ is a left fibration.

The following facts are recorded in [DAGV, Proposition 2.3.5, Proposition 2.3.18].

Proposition 3.4. *Let $\mathcal{G}$ be a geometry.*

(1) $\mathrm{LTop}(\mathcal{G})_{\text{ét}}$ *admits small limits which are preserved by the inclusion* $\mathrm{LTop}(\mathcal{G})_{\text{ét}} \to \mathrm{LTop}(\mathcal{G})$. *Furthermore, an augmented simplicial diagram with values in* $\mathrm{LTop}(\mathcal{G})_{\text{ét}}$ *is a limit diagram if and only if the diagram of underlying topoi in* $\mathrm{LTop}_{\text{ét}}$[5] *is a limit diagram.*

(2) For every $\mathcal{X} \in \mathrm{LTop}(\mathcal{G})$, *one has equivalences* $\mathcal{X}^{\mathrm{op}} \simeq (\mathrm{LTop}(\mathcal{G})_{\text{ét}})_{/(\mathcal{X}', \mathcal{O}')}$ *via* $U \mapsto (\mathcal{X}_{/U}, \mathcal{O}|_U)$.

(3) Given a morphism of geometries $f : \mathcal{G} \to \mathcal{G}'$ *and an étale morphism*

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U) \in \mathrm{LTop}(\mathcal{G})^{[1]}_{\text{ét}}$$

the associated map $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}, \mathcal{O}) \to \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U) \in \mathrm{LTop}(\mathcal{G}')^{[1]}$ *is also étale, and moreover sits in the following co-Cartesian diagram*

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & \longrightarrow & (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U) \\ \downarrow & & \downarrow \\ \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}, \mathcal{O}) & \longrightarrow & \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|_U) \end{array}$$

in LTop, *where the vertical morphisms are associated to the unit transformations of the adjunction* $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} \dashv \mathrm{res}$.

For the following lemma, fix a morphism of geometries $f : \mathcal{G} \to \mathcal{G}'$, and let $(\mathcal{X}, \mathcal{O}) \in \mathrm{LTop}(\mathcal{G})$ be a fixed base.

Lemma 3.5. *The relative spectrum functor* $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}$ *sends limits in* $\mathrm{LTop}(\mathcal{G})_{\text{ét}}$ *to limits in* $\mathrm{LTop}(\mathcal{G}')$.

Proof. We first demonstrate the simpler statement that the relative spectrum sends limits in $(\mathrm{LTop}(\mathcal{G})_{\text{ét}})_{(\mathcal{X}, \mathcal{O})/}$ to limits in $\mathrm{LTop}(\mathcal{G}')$, given $(\mathcal{X}, \mathcal{O}) \in \mathrm{LTop}(\mathcal{G})$. Write $(\mathcal{X}', \mathcal{O}') \coloneqq \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}, \mathcal{O}) \in \mathrm{LTop}(\mathcal{G}')$ and let $\eta^* : \mathcal{X} \to \mathcal{X}' \in \mathrm{LTop}^{[1]}$ denote the induced counit map. Proposition 3.4 implies that the relative spectrum construction lifts to a functor $(\mathrm{LTop}(\mathcal{G})_{\text{ét}})_{(\mathcal{X}, \mathcal{O})/} \to (\mathrm{LTop}(\mathcal{G}')_{\text{ét}})_{(\mathcal{X}', \mathcal{O}')/}$, and the same proposition implies limits in this latter ∞-category may be computed in $\mathrm{LTop}(\mathcal{G}')$. It thus suffices to show that this lift preserves limits. Applying the equivalences of Proposition 3.4.(2) to $(\mathcal{X}, \mathcal{O})$ and $(\mathcal{X}', \mathcal{O}')$ yields a composite of the following form

$$\mathcal{X} \xrightarrow{\sim} (\mathrm{LTop}(\mathcal{G})_{\text{ét}})^{\mathrm{op}}_{/(\mathcal{X}, \mathcal{O})} \to (\mathrm{LTop}(\mathcal{G}')_{\text{ét}})^{\mathrm{op}}_{/(\mathcal{X}', \mathcal{O}')} \xrightarrow{\sim} \mathcal{X}'$$

which may be identified with $\eta^* : \mathcal{X} \to \mathcal{X}'$; as this functor is continuous, the claim follows. Now, given a diagram $p : K \to \mathrm{LTop}(\mathcal{G})_{\text{ét}}$ admitting a limit $p^{\triangleleft} : K^{\triangleleft} \to \mathrm{LTop}(\mathcal{G})_{\text{ét}}$, one has unique lifts of $p, p^{\triangleleft}$ to diagrams $q : K \to (\mathrm{LTop}(\mathcal{G})_{\text{ét}})_{p^{\triangleleft}(\{\infty\})/}$ and similarly for $q^{\triangleleft}$. Since the forgetful functor $(\mathrm{LTop}(\mathcal{G})_{\text{ét}})_{p^{\triangleleft}(\{\infty\})/} \to \mathrm{LTop}(\mathcal{G})_{\text{ét}}$ creates limits, the diagram $q^{\triangleleft}$ must itself be a limit diagram for q. We learn that the composite

$$q^{\triangleleft} : K^{\triangleleft} \to (\mathrm{LTop}(\mathcal{G})_{\text{ét}})_{p^{\triangleleft}(\{\infty\})/} \to \mathrm{LTop}(\mathcal{G}')$$

is a limit diagram over the restriction to K. Since this is naturally identified with $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} \circ p^{\triangleleft}$, the result follows. □

We conclude with the following definition.

[5] or equivalently LTop

Definition 3.6. Given a geometry $\mathcal{G}$, let $(\mathrm{LTop}(\mathcal{G})^{\mathrm{op}}, \mathrm{can})$ denote the (locally large, very large) site where can is the Grothendieck topology consisting of exactly those sieves $\mathcal{C}_{/(\mathcal{X},\mathcal{O})} \subset (\mathrm{LTop}(\mathcal{G})^{\mathrm{op}})_{/(\mathcal{X},\mathcal{O})}$ containing a family of maps $f_i : (\mathcal{X}_i, \mathcal{O}_i) \to (\mathcal{X}, \mathcal{O})$ satisfying:

(1) Each f_i is étale.
(2) Under the equivalence $(\mathrm{LTop}(\mathcal{G})^{\mathrm{op}}_{\text{ét}})_{/(\mathcal{X},\mathcal{O})} \simeq \mathcal{X}$ of Proposition 3.4.(2), f_i corresponds to a family $\{f_i : U_i \to \mathbf{1}\}$ with $\coprod_I U_i \twoheadrightarrow \mathbf{1}$ an effective epimorphism.

Example 3.7. The Yoneda embedding $\text{ℶ} : \mathrm{LTop}(\mathcal{G})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{LTop}(\mathcal{G}), \widehat{\mathcal{S}})$ has essential image contained in $\mathrm{Shv}_{\mathrm{can}}(\mathrm{LTop}(\mathcal{G})^{\mathrm{op}}; \widehat{\mathcal{S}})$.

Example 3.8. Given a morphism of geometries $f : \mathcal{G} \to \mathcal{G}'$, the assignment

$$\{(\mathcal{X}, \mathcal{O}) \mapsto \Gamma_{\mathcal{G}'}(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}, \mathcal{O})\} \in \mathrm{Fun}(\mathrm{LTop}(\mathcal{G}), \mathrm{Ind}(\mathcal{G}'^{\mathrm{op}}))$$

is a sheaf on $(\mathrm{LTop}(\mathcal{G}), \mathrm{can})$. Indeed, let $\mathcal{C}_{/(\mathcal{X},\mathcal{O})}$ be an arbitrary covering sieve. By definition, $\mathcal{C}_{/(\mathcal{X},\mathcal{O})} \hookrightarrow \mathcal{X}_{/1}$ is associated to a covering sieve of $\mathbf{1} \in \mathcal{X}$ in the canonical topology, and thus $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \varprojlim_{\mathcal{C}_{/(\mathcal{X},\mathcal{O})}} (U, \mathcal{O}|_U)$ in $(\mathrm{LTop}(\mathcal{G})_{\text{ét}})_{(\mathcal{X},\mathcal{O})/}$. It follows that the natural map

$$\Gamma_{\mathcal{G}'}(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \xrightarrow{\sim} \varprojlim_{\mathcal{C}_{/(\mathcal{X},\mathcal{O})}} \Gamma_{\mathcal{G}'}(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(U, \mathcal{O}|_U))$$

is an equivalence by the proof of Lemma 3.5 and the fact that $\Gamma_{\mathcal{G}'}$ is a right adjoint, yielding the claim.

3.B. **Localic structured topoi.**

Definition 3.9. Let $\mathcal{X} \in \mathrm{RTop}$. We say $\mathcal{X}$ is 0-*localic* if for any $\mathcal{Y} \in \mathrm{RTop}$ one has an equivalence

$$\mathrm{Fun}^{\mathrm{geom}}(\mathcal{Y}, \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{geom}}(\tau_{\leq -1}\mathcal{Y}, \tau_{\leq -1}\mathcal{X})$$

where $\tau_{\leq -1} \colon \mathrm{RTop} \to \mathrm{Cat}$ sends any ∞-topos to the full subcategory of its (-1)-truncated objects.

Recall that a *frame* is a partially ordered set admitting arbitrary colimits and finite limits, such that finite limits distribute over infinite colimits. We write Loc to denote the category of frames with morphisms given by right adjoints which admit left-exact left adjoints. Given a frame F, we furthermore write $\mathrm{Shv}(F) \coloneqq \mathrm{Fun}^{\mathrm{lim}}(F^{\mathrm{op}}, \mathcal{S})$. The following result is an agglomeration of the results of [HTT, 6.4.2.1, 6.4.5].

Theorem 3.10 (Omnibus 0-localic topoi)**.** *The functor* $\mathrm{Shv}(-) : \mathrm{Loc} \to \mathrm{RTop}$ *is fully faithful with essential image exactly the* 0*-localic topoi. It furthermore admits a left adjoint, given by* $\mathcal{X} \mapsto \tau_{\leq -1}\mathcal{X}$*, whose unit transformation is referred to as the* 0*-localic reflection.*

Notation 3.11. We choose to work in the following settings, mirroring algebro-geometric convention.

(1) Let $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}} \coloneqq \mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})^{\mathrm{op}}$. We refer to this as the ∞-category of *locally spectrally ringed topoi*.
(2) Let $\mathrm{RTop}^{\mathrm{Dir}}_{\mathrm{CAlg}} \coloneqq \mathrm{LTop}(\mathcal{G}_{\mathrm{Dir}})^{\mathrm{op}}$. We refer to this as the ∞-category of *Dirac-locally spectrally ringed topoi*.
(3) Let $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{2CAlg}} \coloneqq \mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})^{\mathrm{op}}$. We refer to this as the ∞-category of *locally 2-ringed topoi*.

In each of the cases above, morphisms $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are *geometric* morphisms $f_* : \mathcal{X} \to \mathcal{Y}$ along with morphisms $\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}$ satisfying a locality condition on their mates.

Example 3.12. There is a full subcategory $\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}} \hookrightarrow \mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})^{\mathrm{op}}$ corresponding exactly to the *locally spectrally-ringed spaces* with hypercomplete sheaves of commutative rings. These have a more familiar definition purely in terms of the pointwise condition of Remark 2.7. We refer the reader to [DAGVII, §2] for details.

Notation 3.13. We will henceforth write $\mathrm{Loc}_{\mathrm{2CAlg}} \coloneqq \mathrm{RTop}_{\mathrm{2CAlg}} \times_{\mathrm{RTop}} \mathrm{Loc}$, and analogously with $\mathrm{Loc}^{\mathrm{loc}}_{\mathrm{2CAlg}}$.

Example 3.14. Given $\mathcal{K} \in \mathrm{2CAlg}$, Theorem 2.27 implies that the underlying ∞-topos of $\mathrm{Spec}\,\mathcal{K}$ is 0-localic and hence $\mathrm{Spec}\,\mathcal{K} \in \mathrm{Loc}^{\mathrm{loc}}_{\mathrm{2CAlg}}$.

3.C. **Relative spectra of spectral schemes are 0-localic.** We recall one equivalent definition of the ∞-category of spectral schemes, following [DAGVII, Definition 2.7].

Definition 3.15. A *spectral scheme*[6], (resp. *Dirac spectral scheme*) is an object $X \in \mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}$ (resp. $\mathrm{RTop}^{\mathrm{Dir}}_{\mathrm{CAlg}}$ satisfying the following two conditions.

(1) The underlying ∞-topos (which we will denote $\mathrm{Shv}(X)$) is 0-localic.
(2) There is an effective epimorphism $\{\coprod U_i \twoheadrightarrow \mathbf{1}\}$ in $\mathrm{Shv}(X)$ such that for every i there exists $R \in \mathrm{CAlg}$ and an equivalence $(\mathrm{Shv}(X)_{/U_i}, \mathcal{O}_X|_{U_i}) \simeq \mathrm{Spec}\,R$ in $\mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})$ (resp. $\mathrm{LTop}(\mathcal{G}_{\mathrm{Dir}})$).

We write $\mathrm{SpSch} \subset \mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}$, (resp. $\mathrm{SpSch}^{\mathrm{Dir}}$) to denote the full subcategory of spectral schemes.

Definition 3.16. We say a spectral scheme is *quasicompact and quasiseparated (qcqs)* if the underlying 0-localic ∞-topos $\mathrm{Shv}(X)$ is coherent in the sense of [DAGVII, §3].

The following result is the main import of this subsection.

Proposition 3.17. *Let* $\mathcal{G} = \mathcal{G}_{\mathrm{cZar}}$ *(resp.* $\mathcal{G}_{\mathrm{Dir}}$*), let* $\mathcal{G}' = \mathcal{G}_{\mathrm{Zar}}$*. Let* $X \in \mathrm{SpSch}, \mathrm{SpSch}^{\mathrm{Dir}}$*. Then* $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X \in \mathrm{Loc}^{\mathrm{loc}}_{\mathrm{2CAlg}}$.

We will need the following establishing lemmas.

Lemma 3.18. *Any left exact functor* $F : \mathcal{C} \to \mathcal{D}$ *between* ∞*-categories admitting finite products*[7] *sends* (-1)*-truncated objects of* $\mathcal{C}$ *to* (-1)*-truncated objects of* $\mathcal{D}$.

Proof. This boils down to the claim that an object $c \in \mathcal{C}$ is (-1)-truncated if and only if the diagonal map $\Delta : c \to c \times c$ is an equivalence, the verification of which we leave to the reader. □

Lemma 3.19. *Let* $f_* : \mathcal{X} \to \mathcal{Y} \in \mathrm{RTop}^{[1]}$ *be a geometric morphism satisfying the following conditions:*

(1) $\mathcal{Y}$ *is 0-localic.*
(2) There is an effective epimorphism $\coprod_I U_i \twoheadrightarrow \mathbf{1} \in \mathcal{Y}^{[1]}$ *such that* $U_i \in \tau_{\leq -1}\mathcal{Y}$ *and* $\mathcal{X}_{/f^* U_i}$ *is* 0*-localic, for every* $i \in I$.

Then $\mathcal{X}$ *is* 0*-localic.*

Proof. Passing to associated left adjoints supplies a factorization of $f^* : \mathcal{Y} \to \mathcal{X}$ through a left-exact left adjoint $\mathcal{Y} \to \mathrm{Shv}(\tau_{\leq -1}\mathcal{X})$ using Theorem 3.10. Using Lemma 3.18 we may now reduce to the case where $\mathcal{Y} \simeq \mathrm{Shv}(\tau_{\leq -1}\mathcal{X})$ and f_* is the reflection $\eta_* : \mathcal{X} \to \mathrm{Shv}(\tau_{\leq -1}\mathcal{X})$. In this case, we are given

[6] these are potentially nonconnective by default.
[7] This condition is easily dropped by passing to presheaf categories.

$\{U_i\}_{i\in I} \subset \tau_{\leq -1}\mathrm{Shv}(\tau_{\leq -1}\mathcal{X}) \simeq \tau_{\leq -1}\mathcal{X}$ along with an effective epimorphism $\coprod_I U_i \twoheadrightarrow \mathbf{1}$ in $\mathcal{X}$. Define the morphisms $f^i_* \in \mathrm{RTop}^{[1]}$ as in the following Cartesian square

$$\begin{array}{ccc} \mathcal{X}_{/U_i} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f^i_*}\downarrow & & \downarrow{\scriptstyle \eta_*} \\ \mathrm{Shv}(\tau_{\leq -1}\mathcal{X})_{/U_i} & \longrightarrow & \mathrm{Shv}(\tau_{\leq -1}\mathcal{X}) \end{array} \tag{3.20}$$

noting that the object $f^{i,*}U_i \in \mathrm{Shv}(\tau_{-1}\mathcal{X})$ may be identified with U_i.

We first claim that the left adjoint to the top horizontal map in (3.20), given by

$$U_i \times - : \mathcal{X} \to \mathcal{X}_{/U_i},$$

induces an equivalence $(\tau_{-1}\mathcal{X})_{/U_i} \simeq \tau_{-1}(\mathcal{X}_{/U_i})$. Since U_i is (-1)-truncated, we have that $x \times_{U_i} y \simeq x \times y$ for any pair of maps $x \to U_i$, $y \to U_i$ in $\mathcal{X}^{[1]}$, owing to the following Cartesian diagram

$$\begin{array}{ccc} x \times_{U_i} x & \longrightarrow & x \times x \\ \downarrow & & \downarrow \\ U_i & = & U_i \times U_i. \end{array}$$

We find that every $x \to U_i \in \mathcal{X}^{[1]}$ satisfies $x \simeq x \times_{U_i} U_i \simeq x \times U_i$, and that any $x \to U_i$ is (-1)-truncated if and only if $x \in \tau_{-1}\mathcal{X}$ from which the claim follows.

The previous claim furnishes an equivalence $\mathrm{Shv}(\tau_{\leq -1}(\mathcal{X}_{/U_i})) \to \mathrm{Shv}(\tau_{\leq -1}\mathcal{X})_{/U_i}$ which moreover identifies f^i_* with the 0-localic reflection $\mathcal{X}_{/U_i} \to \mathrm{Shv}(\tau_{\leq -1}(\mathcal{X}_{/U_i}))$ (by checking that this induces the identity upon application of $\tau_{\leq -1}$). By assumption on U_i, we have that f^i_* is an equivalence for every $i \in I$, whence we have that $\eta_* : \mathcal{X} \to \mathrm{Shv}(\tau_{\leq -1}\mathcal{X})$ is an equivalence upon pulling back along $\mathrm{Shv}(\tau_{\leq -1}\mathcal{X})_{/\coprod_I U_i} \to \mathrm{Shv}(\tau_{\leq -1}\mathcal{X})$. Applying descent [HTT, Theorem 6.1.3.9], we conclude. □

Before we begin the proof of Proposition 3.17, we will also require the following key fact.

Recollection 3.21. Given a transformation of geometries $\mathcal{G} \to \mathcal{G}'$, [DAGV, Proposition 2.3.18.(2)] supplies an identification $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} \mathrm{Spec}^{\mathcal{G}} \simeq \mathrm{Spec}^{\mathcal{G}'}$ of functors $\mathrm{Ind}(\mathcal{G}^{\mathrm{op}}) \to \mathrm{LTop}(\mathcal{G}')$.

Proof of Proposition 3.17. By construction, we are supplied with a counit map $f_* : \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X \to X$ in $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}$. [DAGVII, Theorem 2.40] implies that we may find an effective epimorphism $\coprod_{i\in I} U_i \twoheadrightarrow X$ where $U_i \in \tau_{\leq -1}\mathrm{Shv}(X)$ and $(\mathrm{Shv}(X)_{/U_i}, \mathcal{O}_X|_{U_i}) \simeq \mathrm{Spec}\, A_i$ for $A_i \in \mathrm{CAlg}$; indeed, we may select any cover of the underlying classical scheme by affine opens. Proposition 3.4.(3) now implies that

$$(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X)_{/f^* U_i} \simeq \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} \mathrm{Spec}\, A_i \simeq \mathrm{Spec}\, \mathrm{Perf}_{A_i}$$

the last of which is 0-localic by Example 3.14. We are now in the setting of Lemma 3.19, and we may conclude. □

4. An Affineness Criterion for 2-Schemes

This section is dedicated to the proof of Theorem A, recorded as Theorem 4.7 below. Let us first define our basic objects.

Definition 4.1. An *affine 2-scheme* is an object $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg}}$ which is equivalent to $\mathrm{Spec}\, \mathcal{K}$ for some $\mathcal{K} \in 2\mathrm{CAlg}$. We write $2\mathrm{Aff} \subseteq \mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg}}$ to denote the full subcategory of affine 2-schemes.

Definition 4.2. A *2-scheme* is an object $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg}}$ satisfying the following conditions:

(1) The underlying ∞-topos $\mathcal{X}$ is 0-localic.
(2) There is an effective epimorphism $\{\coprod U_i \twoheadrightarrow \mathbf{1}\}$ in $\mathcal{X}$ such that:
   (a) Each $U_i \in \tau_{\leq -1}\mathcal{X}$
   (b) For every i there exists $\mathcal{K}_i \in 2\mathrm{CAlg}$ and an equivalence $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \simeq \operatorname{Spec} \mathcal{K}_i$ in $\mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$.

We write $2\mathrm{Sch} \subset \mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg}}$ to denote the full subcategory of 2-schemes.

The truncatedness restriction on U_i in part (a) of condition (2) above is meant to ensure that the covering 2-affine neighborhoods of $\mathcal{X}$ are actually associated to subframes of $\tau_{\leq -1}\mathcal{X}$. This at first appears to distinguish it from Definition 3.15. However, condition (2) turns out to be redundant, and we refer the reader to Remark 4.14 at the end of this section for further discussion of the same.

Definition 4.3. We say a 2-scheme is *quasicompact and quasiseparated (qcqs)* if the underlying 0-localic ∞-topos $\mathcal{X}$ is coherent in the sense of [DAGVII, §3].

Example 4.4. By the identification of Theorem 2.27 and the results of [Bal05, §2], one has that $\operatorname{Spec} \mathcal{K} \in 2\mathrm{Sch}_{\mathrm{qcqs}}$ for every $\mathcal{K} \in 2\mathrm{CAlg}$, and hence $2\mathrm{Aff} \subseteq 2\mathrm{Sch}_{\mathrm{qcqs}}$.

Definition 4.5. We say a 2-scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *rigid* if there exists an effective epimorphism $\{\coprod U_i \twoheadrightarrow \mathbf{1}\}$ in $\mathcal{X}$ such that for every i:

(1) $U_i \in \tau_{\leq -1}\mathcal{X}$
(2) There exists $\mathcal{K}_i \in 2\mathrm{CAlg}_{\mathrm{rig}}$ and an equivalence $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \simeq \operatorname{Spec} \mathcal{K}_i$ in $\mathrm{LTop}(\mathcal{G}_{\mathrm{Zar}})$.

Remark 4.6. As one might hope, it is possible to show that the result of Theorem 4.7 implies that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a rigid 2-scheme if and only if *any* étale map

$$\operatorname{Spec} \mathcal{K} \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in (\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}})^{[1]}$$

associated to $U \in \tau_{\leq -1}\mathcal{X}$ factors through a map $f : \operatorname{Spec} \mathcal{K} \to \operatorname{Spec} \mathcal{K}'$ where $\mathcal{K}'$ is rigid and f is an equivalence of locally 2-ringed topoi. Thus, the property of an affine neighborhood to be equivalent to the spectrum of a rigid 2-ring satisfies the "affine commmunication lemma".

The main result of this section is the following.

Theorem 4.7. *A rigid 2-scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is affine if and only if it is qcqs.*

The "only if" direction is just Example 4.4. For the reverse direction, we will need to collect some preliminaries on the behaviour of Karoubi quotients in $\mathrm{Cat}^{\mathrm{perf}}$. We are primarily after Corollary 4.11, which is essentially a symmetric monoidal version and idempotent-complete of [Cal+25, Proposition A.1.18]. In the interest of self-containment we have chosen to include a direct proof that does not assume their result. We will first need the following useful lemma, a weak version of the "second isomorphism theorem" for stable ∞-categories.

Lemma 4.8. *Let $\mathcal{E} \subseteq \mathcal{C} \subseteq \mathcal{D}$ be a sequence of inclusions in $\mathrm{Cat}^{\mathrm{perf}}$. Then the induced functor $\mathcal{C}/\mathcal{E} \to \mathcal{D}/\mathcal{E}$ is fully faithful.*

Proof. Given $x_1, x_2 \in \mathcal{C}$, [NS18, Theorem I.3.3] implies that the map $\operatorname{map}_{\mathcal{C}/\mathcal{E}}(e(x_1), e(x_2)) \to \operatorname{map}_{\mathcal{D}/\mathcal{E}}(f(x_1), f(x_2))$ may be identified with the natural map

$$\varprojlim_{Z \in \mathcal{E}_{\downarrow_{\mathcal{C}} x_2}} \operatorname{map}_{\mathcal{C}}(x_1, \operatorname{cofib}(Z \to x_2)) \to \varprojlim_{Z \in \mathcal{E}_{\downarrow_{\mathcal{D}} x_2}} \operatorname{map}_{\mathcal{C}}(x_1, \operatorname{cofib}(Z \to x_2))$$

where $e : \mathcal{C} \to \mathcal{C}/\mathcal{E}$ and $f : \mathcal{D} \to \mathcal{D}/\mathcal{E}$ are the associated Karoubi quotients. However, since $\mathcal{E} \subseteq \mathcal{C} \subseteq \mathcal{D}$ are inclusions of full subcategories, the induced functor $\mathcal{E} \downarrow_{\mathcal{C}} x_2 \to \mathcal{E} \downarrow_{\mathcal{D}} x_2$ is an equivalence, and for every $Z \in \mathcal{E} \downarrow_{\mathcal{D}} x_2$ the following map is an equivalence

$$\mathrm{map}_{\mathcal{C}}(x_1, \mathrm{cofib}(Z \to x_2)) \xrightarrow{\sim} \mathrm{map}_{\mathcal{D}}(x_1, \mathrm{cofib}(Z \to x_2)).$$

The claim follows. □

Recollection 4.9. We recall the "oriented fiber product" construction of [LT19]. Given a diagram $\mathcal{A} \xrightarrow{p} \mathcal{C} \xleftarrow{q} \mathcal{B}$ in $\mathrm{Cat}^{\mathrm{perf}}$, the oriented fiber product $\mathcal{A} \vec{\times}_{\mathcal{C}} \mathcal{B}$ is defined by the following pullback diagram

$$\begin{array}{ccc} \mathcal{A} \vec{\times}_{\mathcal{C}} \mathcal{B} & \longrightarrow & \mathcal{C}^{[1]} \\ \downarrow & & \downarrow{\scriptstyle s\times t} \\ \mathcal{A} \times \mathcal{B} & \xrightarrow{p\times q} & \mathcal{C} \times \mathcal{C} \end{array}$$

where s, t send a morphism in $\mathcal{C}^{[1]}$ to its source and target respectively. Note that there is a fully faithful inclusion $\mathcal{A} \times_{\mathcal{C}} \mathcal{B} \hookrightarrow \mathcal{A} \vec{\times}_{\mathcal{C}} \mathcal{B}$, with essential image exactly those triples $(x, y, \alpha : p(x) \to q(y))$ such that α is an equivalence in $\mathcal{C}$, using the identification $\mathcal{C} \simeq \mathcal{C}^{[1],\mathrm{eq}}$, where the latter refers to the full subcategory on the equivalences, and the fact that the following is a pullback diagram

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \longrightarrow & \mathcal{C} \simeq \mathcal{C}^{[1],\mathrm{eq}} \\ \downarrow & & \downarrow \\ \mathcal{A} \vec{\times}_{\mathcal{C}} \mathcal{B} & \longrightarrow & \mathcal{C}^{[1]} \end{array}$$

For the next lemma, let $\mathcal{K}_1 \xrightarrow{p} \mathcal{K}_{12} \xleftarrow{q} \mathcal{K}_2$ be a diagram in 2CAlg such that q is identified with a Karoubi quotient of $\mathcal{K}_2$ away from $\mathcal{I}_2 \subset \mathcal{K}_2$.

Lemma 4.10. *The induced functor* $\mathcal{K}_1 \vec{\times}_{\mathcal{K}_{12}} \mathcal{K}_2 \to \mathcal{K}_1 \vec{\times}_{\mathcal{K}_{12}} \mathcal{K}_{12}$ *given by*

$$(x, y, \alpha : p(x) \to q(y)) \mapsto (x, q(y), \alpha : p(x) \to q(y))$$

is a Karoubi quotient with kernel given by the thick subcategory $(0, \mathcal{I}_2, 0)$.

Proof. Let us write $\mathcal{C} := \mathcal{K}_1 \vec{\times}_{\mathcal{K}_{12}} \mathcal{K}_2$, $\mathcal{C}' := \mathcal{K}_1 \vec{\times}_{\mathcal{K}_{12}} \mathcal{K}_{12}$, and $\mathcal{D} := (0, \mathcal{I}_2, 0)$. The identification of the kernel of the map $\mathcal{C} \to \mathcal{C}'$ as in the lemma is clear, and one thus has a conservative functor $\mathcal{C}/\mathcal{D} \to \mathcal{C}'$ which we first claim is fully faithful. By [NS18, Theorem I.3.3], it is equivalent to show that given $(x_1, y_1, \alpha), (x_2, y_2, \beta) \in \mathcal{C}$ arbitrary, the natural map

$$\varinjlim_{Z \in \mathcal{D}_{/(x_2, y_2, \beta)}} \mathrm{map}_{\mathcal{C}}\Big((x_1, y_1, \alpha), \mathrm{cofib}(Z \to (x_2, y_2, \beta))\Big) \to \mathrm{map}_{\mathcal{C}'}((x_1, q(y_1), \alpha), (x_2, q(y_2), \beta))$$

is an equivalence. Using the pullback presentation of $\mathcal{C}'$, the latter mapping space may be expressed via the following Cartesian square

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{C}'}((x_1, q(y_1), \alpha), (x_2, q(y_2), \beta)) & \longrightarrow & \mathrm{map}_{\mathcal{K}_{12}^{[1]}}(\alpha, \beta) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathcal{K}_1}(x_1, x_2) \times \mathrm{map}_{\mathcal{K}_{12}}(q(y_1), q(y_2)) & \longrightarrow & \mathrm{map}_{\mathcal{K}_{12}}(p(x_1), p(x_2)) \times \mathrm{map}_{\mathcal{K}_{12}}(q(y_1), q(y_2)) \end{array}$$

where we note that the following natural map is an equivalence

$$\mathrm{map}_{\mathcal{K}_1}(x_1, x_2) \times \varinjlim_{Z \in \mathcal{I}_{2/y_2}} \mathrm{map}_{\mathcal{K}_{12}}(y_1, \mathrm{cofib}(Z \to y_2)) \to \mathrm{map}_{\mathcal{K}_1}(x_1, x_2) \times \mathrm{map}_{\mathcal{K}_{12}}(q(y_1), q(y_2))$$

since $\mathcal{K}_2 \to \mathcal{K}_{12}$ identifies the target with the Karoubi quotient away from $\mathcal{I}_2$. Using the fact that the target map $t : \mathcal{C} \to \mathcal{K}_2$ is a colocalization with left adjoint given by $y \mapsto (0, y, 0)$, we have that the target map induces an equivalence $\mathcal{D}_{/(x_1,y_2,\beta)} \simeq \mathcal{I}_{2/y_2}$. Altogether, we obtain a Cartesian diagram

$$\begin{array}{ccc} \varinjlim_{Z \in \mathcal{D}_{/(x_2,y_2,\beta)}} \mathrm{map}_{\mathcal{C}}\Big((x_1, y_1, \alpha), \mathrm{cofib}(Z \to (x_2, y_2, \beta))\Big) & \longrightarrow & \mathrm{map}_{\mathcal{C}'}((x_1, q(y_1), \alpha), (x_2, q(y_2), \beta)) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathcal{K}_1}(x_1, x_2) \times \varinjlim_{Z \in \mathcal{I}_{2/y_2}} \mathrm{map}_{\mathcal{K}_{12}}(y_1, \mathrm{cofib}(Z \to y_2)) & \longrightarrow & \mathrm{map}_{\mathcal{K}_1}(x_1, x_2) \times \mathrm{map}_{\mathcal{K}_{12}}(q(y_1), q(y_2)) \end{array}$$

by taking a filtered colimit of the associated Cartesian squares over $Z \in \mathcal{I}_{2/y_2} \simeq \mathcal{D}_{(x_2,y_2,\beta)}$. Since the bottom horizontal arrow is an equivalence, we deduce the claimed fully faithfulness.

It remains to see that the map $\mathcal{C} \to \mathcal{C}'$ has a retract-dense essential image, or equivalently that $\mathcal{C}/\mathcal{D} \hookrightarrow \mathcal{C}'$ is surjective. For this, we note that given any arbitrary $(x, y, \alpha) \in \mathcal{C}'$, the object $(x \oplus \Sigma x, y \oplus \Sigma y, \alpha \oplus \Sigma\alpha)$ sits in the following cofiber sequence:

$$(0, y \oplus \Sigma y, 0) \to (x \oplus \Sigma x, y \oplus \Sigma y, \alpha \oplus \Sigma\alpha) \to (x \oplus \Sigma x, 0, 0)$$

where the third term lives in the essential image of the embedding $\mathcal{K}_1 \hookrightarrow \mathcal{C}'$, and the first term lives in the essential image of the composite $\mathcal{K}_2 \to \mathcal{K}_2/\mathcal{I}_2 \simeq \mathcal{K}_{12} \hookrightarrow \mathcal{C}'$, using the K-theory extension theorem of Neeman-Thomason [Nee92, Corollary 0.9]. Since both of these embeddings factor through the map $\mathcal{C} \to \mathcal{C}'$, it follows that the middle term of the above sequence is in the full subcategory $\mathcal{C}/\mathcal{D} \subseteq \mathcal{C}'$. It follows that (x, y, α) is the retract of an object in $\mathcal{C}/\mathcal{D}$, yielding the claim. □

Corollary 4.11. *Notation as in the previous lemma, one has that the natural map $\mathcal{K}_1 \times_{\mathcal{K}_{12}} \mathcal{K}_2 \to \mathcal{K}_1$ is a Karoubi quotient away from the ideal $(0, \mathcal{I}_2, 0)$. Moreover, the following diagram is co-Cartesian in* 2CAlg*:*

$$\begin{array}{ccc} \mathcal{K}_1 \times_{\mathcal{K}_{12}} \mathcal{K}_2 & \longrightarrow & \mathcal{K}_1 \\ \downarrow & & \downarrow \\ \mathcal{K}_2 & \longrightarrow & \mathcal{K}_{12} \end{array}$$

Proof. It is clear that the following composite

$$\mathcal{K}_1 \times_{\mathcal{K}_{12}} \mathcal{K}_2 \hookrightarrow \mathcal{K}_1 \vec{\times}_{\mathcal{K}_{12}} \mathcal{K}_2 \to \mathcal{K}_1 \vec{\times}_{\mathcal{K}_{12}} \mathcal{K}_{12}$$

has essential image contained in the full subcategory of triples (x, y, α) such that $\alpha \in \mathcal{K}_{12}^{[1],\mathrm{eq}}$ which by Recollection 4.9 corresponds exactly to the image of $\mathcal{K}_1 \times_{\mathcal{K}_{12}} \mathcal{K}_{12} \subseteq \mathcal{K}_1 \vec{\times}_{\mathcal{K}_{12}} \mathcal{K}_{12}$. Applying Lemma 4.10 and Lemma 4.8, one obtains a fully faithful inclusion $(\mathcal{K}_1 \times_{\mathcal{K}_{12}} \mathcal{K}_2)/(0, \mathcal{I}_2, 0) \hookrightarrow \mathcal{K}_1$, and it remains to show that this is essentially surjective. This uses the same argument as in the last paragraph of Lemma 4.10.

The fact that the square is co-Cartesian follows from [HZG, Corollary 2.30], which states that the following square is co-Cartesian in 2CAlg:

$$\begin{array}{ccc} \mathcal{K}_1 \times_{\mathcal{K}_{12}} \mathcal{K}_2 & \longrightarrow & \mathcal{K}_1 \\ \downarrow & & \downarrow \\ \mathcal{K}_2 & \longrightarrow & \mathcal{K}_2/(0, \mathcal{I}_2, 0) \end{array}$$

where we have also written $(0, \mathcal{I}_2, 0)$ to refer to its image in $\mathcal{K}_2$. However, this is exactly $\mathcal{I}_2 \subseteq \mathcal{K}_2$, which identifies the bottom horizontal arrow above with the localization map $\mathcal{K}_2 \to \mathcal{K}_{12}$. □

Proof of Theorem 4.7. We will induct on the number of affines required to cover a rigid qcqs 2-scheme. The case $n = 1$ is trivial. For the inductive case, let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ admit a cover by affine 2-subschemes $U_1, ..., U_n$, where each $U_i \simeq \operatorname{Spec} \mathcal{K}_i$ for $\mathcal{K}_i \in 2\mathrm{CAlg}_{\mathrm{rig}}$. Let $U := |(\coprod_{i=1}^{n-1} U_i)^{\times \bullet}| \in \mathcal{X}$, and consider the scheme $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_U)$[8]. By the inductive hypothesis, $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|_\mathcal{U}) \simeq \operatorname{Spec} \mathcal{L}$ where $\mathcal{L} := \mathcal{O}(U) \in 2\mathrm{CAlg}_{\mathrm{rig}}$. By the assumptions of coherence of $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and Theorem 2.27, one has that the object $U \times U_n \in \tau_{\leq -1}\mathcal{X}_{/U_n} \simeq \tau_{\leq -1} \operatorname{Spec} \mathcal{K}_n$ corresponds to a quasicompact open of the Balmer spectrum. By the same theorem and rigidity, the map $(\mathcal{X}_{/U\times U_n}, \mathcal{O}_X|_{U\times U_n}) \hookrightarrow (\mathcal{X}_{/U_n}, \mathcal{O}_X|_{U_n})$ may be canonically identified with the map $\operatorname{Spec} \mathcal{K}_n/\langle a \rangle \to \operatorname{Spec} \mathcal{K}_n$ associated to the Karoubi quotient of $\mathcal{K}_n$ by some $a \in \mathcal{K}_n$. Similarly, one may identify the map $(\mathcal{X}_{/U\times U_n}, \mathcal{O}_\mathcal{X}|_{U\times U_n}) \hookrightarrow (\mathcal{X}_{/U}, \mathcal{O}_X|_U)$ with the localization map associated to some $b \in \mathcal{L}$. Using that $U \coprod_{U\times U_n} U_n \simeq \mathbf{1} \in \mathcal{X}$ and Proposition 3.4, we have the following co-Cartesian square in $\mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg}}$:

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{L}/\langle b \rangle \simeq \operatorname{Spec} \mathcal{K}_n/\langle a \rangle & \longrightarrow & \operatorname{Spec} \mathcal{K}_n \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{L} & \longrightarrow & (\mathcal{X}, \mathcal{O}_\mathcal{X}) \end{array} \tag{4.12}$$

where all morphisms are étale. Passing to global sections and applying Corollary 4.11, one obtains the following Cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{K}_n/\langle a \rangle \simeq \operatorname{Spec} \mathcal{L}/\langle b \rangle & \longrightarrow & \operatorname{Spec} \mathcal{K}_n \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{L} & \longrightarrow & \operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \end{array} \tag{4.13}$$

where the vertical and horizontal morphisms are each associated to Karoubi quotients of $\operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$, thus every morphism is étale.

Under the equivalence of categories $(\mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg},\text{ét}})_{/\operatorname{Spec} \Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})} \simeq \operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ of Proposition 3.4.(2), the étale map $\operatorname{Spec} \mathcal{L} \times \operatorname{Spec} \mathcal{K}_n \to \operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ corresponds to an effective epimorphism $U \coprod V \twoheadrightarrow \mathbf{1}$ and hence an identification $\mathbf{1} \simeq U \coprod_{U\times V} V$. Unwinding the equivalences and applying Proposition 3.4.(1), we find that the Cartesian diagram of (4.13) is in fact co-Cartesian as well. There is an "affinization map" from the square of (4.12) to that of (4.13) induced by the unit of the adjunction $\Gamma \dashv \operatorname{Spec}$. As this is the identity on every vertex except the bottom right, we find that the affinization map $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to \operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an equivalence, yielding the inductive case and hence the claim. □

As promised, we elaborate on the apparent difference between Definition 3.15 and Definition 4.2 before concluding.

Remark 4.14. In Definition 4.2 the restriction to truncated objects in part (a) of condition (2) is actually unnecessary, as we now explain.

First we claim that even with just condition (1) and part (b) of condition (2), the underlying frame $\tau_{\leq -1}\mathcal{X}$ can be shown to be *spatial*, i.e., it is the frame of open subsets of a topological space. Write $\mathrm{Frm} \simeq \mathrm{Loc}^{\mathrm{op}}$. Using part (b) of condition (2), the assumption that $\{\coprod U_i \to \mathbf{1}\}$ is an effective epimorphism, we obtain the following equalizer diagram in Frm:

$$\tau_{\leq -1}\mathcal{X} \to \prod \tau_{\leq -1}(\mathcal{X}_{/U_i}) \rightrightarrows \prod \tau_{\leq -1}(\mathcal{X}_{/U_i \times U_j}).$$

[8] spatially, this corresponds to the union of the open subschemes $U_1, ..., U_{n-1}$.

Recall that $\tau_{\leq -1}\mathcal{X}_{/U_i}$ is associated to the frame of opens of the topological space $|\operatorname{Spec}\mathcal{K}_i| \simeq \operatorname{Spc}\mathcal{K}_i$ by Theorem 2.27. The claim now follows from the fact that Stone Duality implies spatial frames are closed under limits ([Joh86, §II.3], see also [HZG, Recollection 4.8]).

Keeping the conditions as above, recall that the maps $\operatorname{Spec}\mathcal{K}_i \to \mathcal{X} \in \mathrm{RTop}^{[1]}$ are étale maps of ∞-topoi. In fact, by [DAGV, Lemma 2.3.16] the objects U_i are in $\tau_{\leq 0}\mathcal{X}$ since $\mathcal{X}$ is 0-localic and $\mathcal{X}_{/U_i} \simeq \operatorname{Spec}\mathcal{K}_i$ is 0-localic. Restricting to underlying ordinary topoi yields a map

$$\mathrm{Shv}(|\operatorname{Spec}\mathcal{K}_i|;\mathrm{Sets}) \simeq \mathrm{Shv}(|\tau_{\leq -1}\mathcal{X}|;\mathrm{Sets})_{/U_i} \to \mathrm{Shv}(|\tau_{\leq -1}\mathcal{X}|;\mathrm{Sets})$$

from which it follows that underlying morphism of spaces $\phi : |\operatorname{Spec}\mathcal{K}_i| \to |\tau_{\leq -1}\mathcal{X}|$ is a local homeomorphism, associated to the *espace étalé* of the sheaf of sets U_i on $|\tau_{\leq -1}\mathcal{X}|$.

Given any point $x \in |\tau_{\leq -1}\mathcal{X}|$ in the image of ϕ, x is contained in an open neighborhood U which is homeomorphic via the map above to an open neighborhood $U(a) \subseteq |\operatorname{Spec}\mathcal{K}_i|$, as these form a basis for the Balmer spectrum. Moreover, since this is an étale map of *locally* 2*-ringed topoi*, we may deduce that ϕ induces equivalences

$$(\mathcal{X}_{/U}, \mathcal{O}|_U) \simeq (\mathrm{Shv}(U), \mathcal{O}|_U) \simeq U(a) \simeq \operatorname{Spec}\mathcal{K}_i/\langle a\rangle$$

of locally 2-ringed topoi, where by construction $U \in \tau_{\leq -1}\mathcal{X}$. Selecting such neighborhoods around every point $x \in |\tau_{\leq -1}\mathcal{X}|$ gives rise to a collection of objects in $\tau_{\leq -1}\mathcal{X}$ satisfying the desired parts (a) and (b) of condition (2) in Definition 4.2.

5. Affineness and Reconstruction of Schemes

5.A. **Relative affineness of schemes.** In this section we demonstrate that the relative spectra of qcqs (Dirac) spectral schemes are affine 2-schemes. To identify their global sections, we will need to introduce the 2-ring of perfect complexes on a spectral or Dirac spectral scheme.

Definition 5.1. As per our convention, we write $\widehat{2\mathrm{CAlg}}$ to denote $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{perf}})$, the very large ∞-category of large 2-rings. Passing to a larger universe, [SAG, Construction 6.2.1.7] constructs an extension of the functor $\mathrm{Perf} : \mathrm{CAlg} \to 2\mathrm{CAlg} \subseteq \widehat{2\mathrm{CAlg}}$ to a limit-preserving to a functor $\mathrm{Perf} : \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{2\mathrm{CAlg}}$.

Lemma 5.2. *Consider the functor-of-points map*

$$\mathrm{SpSch} \subseteq \mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}} \xrightarrow{\text{ⓨ}} \mathrm{Fun}(\mathrm{RTop}^{\mathrm{loc,op}}_{\mathrm{CAlg}}, \widehat{\mathcal{S}}) \xrightarrow{-\circ\mathrm{Spec}} \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}}).$$

The functor Perf *of Definition 5.1 takes values in* $2\mathrm{CAlg}_{\mathrm{rig}} \subseteq \widehat{2\mathrm{CAlg}}$ *when evaluated against any object in the essential image of* SpSch*. The same holds if* SpSch *and* $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}$ *are replaced by* $\mathrm{SpSch}^{\mathrm{Dir}}$ *and* $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{Dir}}$.

Proof. [DAGV, Theorem 2.4.1] implies that the essential image of the functor-of-points factors through $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$. In particular, any object $X \in \mathrm{SpSch}$ has essential image given by a small colimit of the form in $\mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$. In particular, as Perf is limit-preserving, the value of Perf on X may be presented as a small limit in 2CAlg of objects of the form Perf_R for $R \in \mathrm{CAlg}$. As these latter objects are small and rigid, and the inclusion $2\mathrm{CAlg}_{\mathrm{rig}} \subseteq \widehat{2\mathrm{CAlg}}$ is closed under small limits, the result follows. The same argument holds in the Dirac case. □

Henceforth, the assignment Perf constructed above from SpSch (resp. $\mathrm{SpSch}^{\mathrm{Dir}}$) to the ∞-category $2\mathrm{CAlg}_{\mathrm{rig}}$ is referred to as the functor of *perfect complexes*. With this definition in tow, we are ready to prove the main result of this section, Theorem B.

Theorem 5.3. *Let* $\mathcal{G} = \mathcal{G}_{\mathrm{cZar}}$ *(resp.* $\mathcal{G}_{\mathrm{Dir}}$*) and let* $\mathcal{G}' = \mathcal{G}_{\mathrm{Zar}}$*. Then for any* $X \in \mathrm{SpSch}_{\mathrm{qcqs}}$ *(resp.* $\mathrm{SpSch}^{\mathrm{Dir}}_{\mathrm{qcqs}}$*) one has a natural identification*

$$\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X \simeq \mathrm{Spec}\,\mathrm{Perf}_X$$

Proof. Proposition 3.17 and Proposition 3.4.(2) imply that $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X$ is a qcqs 2-scheme. Consider any collection of objects $U_i \in \tau_{\leq -1}\mathrm{Shv}(X)$ which jointly cover $\mathbf{1} \in \mathrm{Shv}(X)$ and satisfy $(\mathrm{Shv}(X)_{/U_i}, \mathcal{O}_X|_{U_i}) \simeq \mathrm{Spec}\, R_i$. Proposition 3.4.(3) sends this to an étale cover of $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X$ by objects of the form $\mathrm{Spec}\,\mathrm{Perf}_{R_i}$, each of which is associated to a (-1)-truncated object of the underlying ∞-topos of $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X$. As these are rigid by Example 2.16, the relative spectrum is a rigid qcqs 2-scheme. By Theorem 4.7, it follows that the relative spectrum is affine, and the result will follow if we can supply natural identification

$$\Gamma(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X, \mathcal{O}) \simeq \mathrm{Perf}_X,$$

which is done below in Proposition 5.8. □

We demonstrate the missing proposition (Proposition 5.8) required to complete the proof of Theorem 4.7.

Construction 5.4. Consider the following diagram of left adjoints:

$$\begin{array}{ccc} \mathrm{CAlg} & \xrightarrow{\mathrm{Spec}} & \mathrm{LTop}(\mathcal{G}) \\ {\scriptstyle \mathrm{Perf}}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}} \\ 2\mathrm{CAlg} & \xrightarrow[\mathrm{Spec}]{} & \mathrm{LTop}(\mathcal{G}') \end{array} \tag{5.5}$$

where $\alpha \in \mathrm{Fun}(\mathrm{CAlg}, \mathrm{LTop}(\mathcal{G}'))^{[1]}$ is the natural equivalence of Recollection 3.21; note that we have dropped the superscripts from the horizontal functors and leave the geometries implicit. Using the adjunction of Theorem 2.12 and passing to horizontal mates, we obtain a comparison morphism:

$$\alpha^h : \mathrm{Perf}_{\Gamma(-,\mathcal{O})} \Rightarrow \Gamma(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(-), \mathcal{O}) \in \mathrm{Fun}(\mathrm{LTop}(\mathcal{G}), 2\mathrm{CAlg})^{[1]} \tag{5.6}$$

Definition 5.7. We refer to the admissible topologies on $\mathrm{CAlg}^{\mathrm{op}}$ under the identifications $\mathrm{CAlg}^{\mathrm{op}} \simeq \mathrm{Pro}(\mathcal{G}_{\mathrm{cZar}})$, $\mathrm{CAlg}^{\mathrm{op}} \simeq \mathrm{Pro}(\mathcal{G}_{\mathrm{Dir}})$ as the *Zariski* and *Dirac* topologies respectively. Here the notion of admissible is as in [DAGV, Notation 2.2.2]: For a given geometry $\mathcal{G}$, recall that an *admissible* morphism $f : U \to X$ for $U, X \in \mathrm{Pro}(\mathcal{G})$ is one for which there exists a pullback diagram as follows

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \downarrow & & \downarrow{\scriptstyle f'} \\ X & \longrightarrow & X' \end{array}$$

where $U', X' \in \mathcal{G}$ and f' is an admissible morphism for $\mathcal{G}$.

The next proposition will explicitly identify α^h and its target using Zariski (resp. Dirac) descent, thereby concluding this subsection.

Proposition 5.8. *The inclusion*

$$\mathrm{Shv}_{\mathrm{can}}(\mathrm{SpSch}; \widehat{2\mathrm{CAlg}}) \subseteq \mathrm{Fun}(\mathrm{SpSch}^{\mathrm{op}}, \widehat{2\mathrm{CAlg}})$$

admits a left adjoint localization. We refer to both this localization and its unit transformation as sheafification with respect to the canonical topology. Keeping the notation of Construction 5.4, we have:

(1) *The map* $\alpha^h \in \mathrm{Fun}(\mathrm{SpSch}^{\mathrm{op}}, 2\mathrm{CAlg})^{[1]}$ *is identified with the sheafification of the source with respect to the canonical topology.*
(2) *The sheafification of* $\mathrm{Perf}_{\Gamma(-,\mathcal{O})}$ *with respect to the canonical topology on* SpSch *is given by the assignment* $X \mapsto \mathrm{Perf}_X$.

The same results hold if SpSch *is replaced by* $\mathrm{SpSch}^{\mathrm{Dir}}$.

Proof. Our first task is to construct the desired left adjoint localization. For our purposes, we will need to derive an explicit formula in terms of Zariski sheaves on $\mathrm{CAlg}^{\mathrm{op}}$. Let $\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ denote the ∞-category of Zariski sheaves on $\mathrm{CAlg}^{\mathrm{op}}$ valued in $\widehat{\mathcal{S}}$. Recall that the inclusion

$$\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}) \subseteq \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$$

admits a left-exact left adjoint localization by applying [HTT, Lemma 6.2.2.7] in a larger universe. The functor of points embedding [DAGV, Theorem 2.4.1] states that the following composite is fully faithful

$$\mathrm{SpSch} \xrightarrow{\text{よ}} \mathrm{Fun}(\mathrm{SpSch}, \widehat{\mathcal{S}}) \xrightarrow{-\circ\mathrm{Spec}} \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}}) \to \widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$$

and furthermore sends colimits in $\mathrm{SpSch}_{\text{ét}} \subseteq \mathrm{LTop}(\mathcal{G}_{\mathrm{cZar}})_{\text{ét}}$ to colimits in $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg})$. Thus, for any very large ∞-category $\mathcal{C}$ admitting all large colimits, the induced right adjoint functor $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \mathcal{C}) \to \mathrm{Fun}(\mathrm{SpSch}, \mathcal{C})$ lifts to $\mathrm{Shv}_{\mathrm{can}}(\mathrm{SpSch}; \mathcal{C})$ using the same argument as in Example 3.8. This gives rise to the following adjunction

$$\iota^* : \mathrm{Shv}_{\mathrm{can}}(\mathrm{SpSch}; \mathcal{C}) \rightleftarrows \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \mathcal{C}) : \iota_*$$

here, ι^* is given by restricting a sheaf along the map of sites $\mathrm{Spec} : (\mathrm{CAlg}^{\mathrm{op}}, \mathrm{Zar}) \hookrightarrow (\mathrm{SpSch}, \mathrm{can})$. To describe ι_*, note that there is an equivalence

$$\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{R}}(\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{C})$$

which is inverse to the restriction from the right hand side to the left hand side, see [SAG, Proposition 1.3.1.7]. Under this equivalence, ι_* sends a sheaf $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \mathcal{C})$ to the composite

$$\mathrm{SpSch}^{\mathrm{op}} \xrightarrow{\text{よ}} \widehat{\mathrm{Shv}}_{\mathrm{can}}(\mathrm{SpSch})^{\mathrm{op}} \to \widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathcal{C}.$$

Now recall that by the subcanonicality of the Zariski topology, the functor

$$\mathrm{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{SpSch} \to \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})^{\mathrm{op}}$$

may be identified with the Yoneda embedding. From this, it is easy to directly compute that the counit $\iota^*\iota_* \xrightarrow{\sim} \mathrm{id}$ as endofunctors of $\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$. Additionally, ι^* is conservative, owing to the following two facts:

- Every quasi-affine spectral scheme X admits a canonical cover by affines $\{\coprod \mathrm{Spec}\, R_i \twoheadrightarrow X\}$ with all terms in the associated Čech complex affine.
- Every object $X \in \mathrm{SpSch}$ admits a canonical cover by affines with all terms in the associated Čech complex quasi-affine.

It follows that the restriction ι^* is conservative, as any morphism $\mathcal{F} \to \mathcal{G} \in \mathrm{Shv}_{\mathrm{can}}(\mathrm{SpSch})^{[1]}$ inverted by ι^* is by definition an equivalence when evaluated on any affine, hence on any quasi-affine, and therefore on arbitrary spectral schemes. We deduce that the pair $\iota^* \dashv \iota_*$ supply mutually

inverse equivalences. Let us now demonstrate claim (1). Note first that there is an equivalence $2\mathrm{CAlg} \simeq \mathrm{Fun}^{\mathrm{lex}}(2\mathrm{CAlg}^{\omega,\mathrm{op}}, \mathcal{S})$. This induces a composite equivalence

$$\begin{aligned} \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \widehat{2\mathrm{CAlg}}) &\simeq \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \widehat{2\mathrm{CAlg}}) \\ &\simeq \mathrm{Fun}^{\mathrm{R}}(\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}), \mathrm{Fun}^{\mathrm{lex}}(\mathrm{CAlg}^{\omega,\mathrm{op}}, \widehat{\mathcal{S}})) \\ &\simeq \mathrm{Fun}^{\mathrm{lex,R}}(\mathrm{CAlg}^{\omega,\mathrm{op}} \times \widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}), \widehat{\mathcal{S}}) \\ &\simeq \mathrm{Fun}^{\mathrm{lex}}(\mathrm{CAlg}^{\omega,\mathrm{op}}, \mathrm{Fun}^{\mathrm{R}}(\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}), \widehat{\mathcal{S}})) \\ &\simeq \mathrm{Fun}^{\mathrm{lex}}(\mathrm{CAlg}^{\omega,\mathrm{op}}, \widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})) \end{aligned} \tag{5.9}$$

which also holds if $\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ (resp. for target $\widehat{2\mathrm{CAlg}}$) is replaced with $\mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$ (resp. for target $\widehat{2\mathrm{CAlg}}$). Moreover, these equivalences sit in a commutative square of adjoints:

$$\begin{array}{ccc} \mathrm{Fun}(\mathrm{CAlg}, \widehat{2\mathrm{CAlg}}) & \rightleftarrows & \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}; \widehat{2\mathrm{CAlg}}) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Fun}^{\mathrm{lex}}(\mathrm{CAlg}^{\omega,\mathrm{op}}, \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})) & \rightleftarrows & \mathrm{Fun}^{\mathrm{lex}}(\mathrm{CAlg}^{\omega,\mathrm{op}}, \widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})) \end{array}$$

see for example the proof of [HZG, Lemma 3.27]. In particular, there is a left adjoint *sheafification with respect to the Zariski topology*

$$\mathrm{Fun}(\mathrm{CAlg}, \widehat{2\mathrm{CAlg}}) \to \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \widehat{2\mathrm{CAlg}})$$

whose right adjoint is the natural forgetful inclusion. The desired sheafification with respect to the *canonical* topology is then given by the composite

$$\mathrm{Fun}(\mathrm{SpSch}^{\mathrm{op}}, \widehat{2\mathrm{CAlg}}) \to \mathrm{Fun}(\mathrm{CAlg}, \widehat{2\mathrm{CAlg}}) \to \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \widehat{2\mathrm{CAlg}}) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{can}}(\mathrm{SpSch}; \widehat{2\mathrm{CAlg}}).$$

To demonstrate claim (1), recall that the map α^h is an equivalence on affine schemes by Recollection 3.21 and thus it must be sent to an equivalence after sheafification with respect to the canonical topology, using the previous paragraph. Example 3.8 shows that it is already the case that

$$\Gamma(\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}_{\mathcal{G}_{\mathrm{cZar}}}(-), \mathcal{O}) \in \mathrm{Shv}_{\mathrm{can}}(\mathrm{SpSch}; 2\mathrm{CAlg})$$

and hence α^h must be identified with the sheafification of its source.

To conclude, it remains to identify the image of $\mathrm{Perf}_{\Gamma(-,\mathcal{O})}$ under sheafification with respect to the canonical topology. By construction, this is given by the composite

$$\mathrm{SpSch} \hookrightarrow \widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg})^{\mathrm{op}} \xrightarrow{\mathcal{P}erf} \widehat{2\mathrm{CAlg}}$$

where $\mathcal{P}erf$ is the image of the functor $\mathrm{Perf} \in \mathrm{Fun}(\mathrm{CAlg}, \widehat{2\mathrm{CAlg}})$ under the composite

$$\mathrm{Fun}(\mathrm{CAlg}, \widehat{2\mathrm{CAlg}}) \to \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}; \widehat{2\mathrm{CAlg}}) \simeq \mathrm{Fun}^{\mathrm{R}}(\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})^{\mathrm{op}}, \widehat{2\mathrm{CAlg}}).$$

Recall that the functor Perf already satisfies Zariski descent on $\mathrm{CAlg}^{\mathrm{op}}$, which follows from the fact that Mod satisfies Zariski descent [SAG, Corollary D.6.3.3] and the fact that passage to subcategories of dualizable objects is a limit preserving functor. Using the argument of [SAG, Proposition 6.2.3.1], we find that the functor $\mathrm{Perf} : \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})^{\mathrm{op}} \to \widehat{2\mathrm{CAlg}}$ factors through the sheafification with respect to the Zariski topology, and in particular that $\mathcal{P}erf \simeq \mathrm{Perf}$ when regarded as objects of $\mathrm{Fun}(\mathrm{CAlg}, \widehat{2\mathrm{CAlg}})$. It follows that $\mathcal{P}erf$ can be identified with the composite

$$\widehat{\mathrm{Shv}}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})^{\mathrm{op}} \xrightarrow{\mathrm{Perf}} \widehat{2\mathrm{CAlg}}$$

and thus the image of $\mathrm{Perf}_{\Gamma(-,\mathcal{O})}$ under the sheafification with respect to the canonical topology on SpSch may be definitionally identified with the functor of perfect complexes.

Finally, the only missing component for running the arguments above in the Dirac case is the fact that the functor $\mathrm{Perf} : \mathrm{CAlg} \to 2\mathrm{CAlg}$ in fact satisfies Dirac descent. For a given $R \in \mathrm{CAlg}$, the comparison map

$$\mathrm{Spec}^{\mathcal{G}_{\mathrm{Zar}}}_{\mathcal{G}_{\mathrm{Dir}}} \mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}} R \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}} R$$

identifies the pushforward of the structure sheaf on the source with the unique assignment on $|\mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}} R|$ satisfying $\{D(f) \subseteq |\mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}} R|\} \mapsto \mathrm{Perf}_{R[f^{-1}]}$ where $D(f)$ is a basic quasicompact open subset of $|\mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}} R|$, see [HZG, Section 3.E] for the structure theory and [HZG, Theorem 4.48] for the identification of sheaves. Since this is in fact a sheaf, we may conclude that Perf satisfies descent for Dirac covers of R, and hence that Perf satisfies Dirac descent on $\mathrm{CAlg}^{\mathrm{op}}$. □

5.B. **A reconstruction theorem of Balmer.** We now obtain Theorem D as a corollary of Theorem 4.7.

Corollary 5.10. *There is a natural comparison transformation*

$$\gamma : (\mathrm{Spec}\,\mathrm{Perf}_{(-)}, \mathcal{R}_{\mathcal{O}_{\mathrm{Perf}_{(-)}}}) \to \mathrm{id}$$

of functors from SpSch *to* $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}$. *For a X an ordinary qcqs scheme, regarded as a 0-truncated spectral scheme, the comparison map γ evaluated on X is an equivalence.*

Proof. From the natural identification of Theorem 4.7, it suffices to show the result with $\mathrm{Spec}\,\mathrm{Perf}_{(-)}$ replaced by $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}$. Since the relative spectrum is left adjoint to restriction, one obtains a counit transformation

$$(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}, \mathcal{R}_{\mathcal{O}}) \to \mathrm{id} \in \mathrm{Fun}(\mathrm{SpSch}, \mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}})^{[1]}$$

yielding the first part of the result. For the second, note that for any cover by affine open subsets $\bigcup_I \mathrm{Spec}\, R_i \subseteq X$, one has an identification

$$(\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} X, \mathcal{R}_{\mathcal{O}}) \times_X \coprod_I \mathrm{Spec}\, R_i \simeq \coprod_I (\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} \mathrm{Spec}\, R_i, \mathcal{R}_{\mathcal{O}_i}) \simeq \coprod_I (\mathrm{Spec}\,\mathrm{Perf}_{R_i}, \mathcal{R}_{\mathcal{O}_i})$$

from Proposition 3.4.(3). We thus reduce to the case where $X = \mathrm{Spec}\, R$ and may furthermore assume R is a classical Noetherian ring by using that $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}} : \mathrm{CAlg} \to \mathrm{LTop}(\mathcal{G}')$ is a left adjoint, and that the restriction $\mathrm{LTop}(\mathcal{G}') \to \mathrm{LTop}(\mathcal{G})$ preserves filtered colimits [DAGV, Corollary 1.5.4]. This case is due to Neeman [NB92], see for example [HZG, Theorem 4.48]. □

Warning 5.11. Note that the proof above recovers the ordinary scheme X regarded as an object of SpSch and not simply as an ordinary ringed space. In particular, the sections of the structure sheaf on any given open set may not be concentrated in π_0[9]. The classically ringed space associated to X may be recovered by taking the sheafification of the assignment $U \mapsto \pi_0 \mathcal{O}_X \in \mathrm{CAlg}^{\heartsuit}$, which is the context originally considered in [Bal02].

Remark 5.12. The previous strategy of proof demonstrates that for X a qcqs spectral scheme, the underlying set of the Zariski spectrum stratifies into:

(1) A "geometric" direction, corresponding to the support theory of the underlying scheme itself.
(2) A "homotopical" direction, corresponding to the failure of the comparison maps $|\mathrm{Spec}\,\mathrm{Perf}_{\mathcal{O}_{X,x}}| \to |\mathrm{Spec}\,\mathcal{O}_{X,x}|$ to be injective or surjective at points $x \in X$.

[9]The example $X = \mathbb{A}^1 \setminus 0$ supplies a case where $\pi_{-1}\Gamma(X, \mathcal{O}) =: H^1(X, \mathcal{O})$ is nontrivial.

For example, the result of Corollary 5.10 holds for regular, noetherian, locally even periodic schemes (for example, an oriented elliptic curve over a regular noetherian even periodic $\mathbb{E}_\infty$-ring) by reduction to the case of regular noetherian even periodic $\mathbb{E}_\infty$-ring spectra, which is handled in [Mat15a, §2].

Remark 5.13. It is often the case that there are interesting specializations between points in distinct fibers of the comparison map $|\operatorname{Spec}\operatorname{Perf}_X| \to |X|$: see, for example, the "blueshifting" behaviour exhibited in the fibers of the map $|\operatorname{Spec}\operatorname{Perf}_{\mathbb{S}^G_G}| \to |\operatorname{Spec}\pi_0\mathbb{S}^G_G|$ where G is a finite group and $\mathbb{S}^G_G$ refers to the categorical fixed points of the genuine G-sphere [San25, §13].

Finally, let us mention the differences between our approach to the computation of $|\operatorname{Spec}\operatorname{Perf}_X|$ for an ordinary qcqs scheme X and the approach adopted by Thomason in [Tho97].

Remark 5.14. We briefly recall the components of the argument in [Tho97], which is a classification of the thick tensor ideals of Perf_X. By the identification between the underlying space of $\operatorname{Spec}\operatorname{Perf}_X$ and the Balmer spectrum $\operatorname{Spc}\operatorname{Perf}_X$, this is equivalent to the desired computation, see for example [HZG, §4.2] for a quick overview of the same.

First, let X be a noetherian qcqs potentially non-affine scheme. The classification of thick tensor ideals consists of two steps:

(1) Demonstrating that there are objects of Perf_X whose cohomology groups are supported on any choice of closed subscheme $Z \subseteq X$.
(2) Showing that any thick tensor ideal $\mathcal{I} \subseteq \operatorname{Perf}_X$ is determined by the union of the supports of cohomology groups of complexes in $\mathcal{I}$.

The second component follows via a nilpotence theorem for Perf_X, extending the results of [Hop87] and [NB92]. As both his and our approaches rely on these results in the affine case, the distinctions here are at most cosmetic.

In order to demonstrate the first component, the author invokes a K-theory extension theorem to inductively build complexes with the desired support by inducting from affines, see [KP17, Lemma 4.1.8] for a brief overview of his method along with a systematization using the "Reduction Principle" of [BV03]. By constrast, we do not construct complexes supported on every possible closed subset of X. In fact, the comparison transformation $\gamma_X : \operatorname{Spec}\operatorname{Perf}_X \to X$ often fails to be surjective for a general spectral scheme, owing to the fact that complexes with prescribed support may not always be constructed. For an ordinary scheme, the existence of these complexes is deduced *a posteriori* from the construction of a functorial comparison map and a direct reduction to the affine case.[10]

6. Geometrization of 2-rings

This section is dedicated to the proof of Theorem E, recorded as Theorem 6.6 below.

6.A. **An aside on unigenicity.**

Before proceeding, it will be helpful later to systematize the exact relation between rings and 2-rings. Recall the following notion, discussed in [HPS97] and [San25, §8].

Definition 6.1. $\mathcal{K} \in \mathrm{2CAlg}$ is *unigenic* if the smallest thick subcategory containing $\mathbf{1}_{\mathcal{K}}$ is $\mathcal{K}$ itself. We write $\mathrm{2CAlg}^{\mathrm{uni}} \subset \mathrm{2CAlg}$ to denote the full subcategory of unigenic 2-rings.

Recollection 6.2. Recall the adjunction of Definition 2.24

$$\operatorname{Perf} : \mathrm{CAlg} \rightleftarrows \mathrm{2CAlg} : \mathcal{R}_{(-)}$$

[10] In some sense, this too is almost cosmetic: the construction of requisite complexes is hidden in the iterative process of Theorem 4.7, but we will not expand upon this here.

whose left adjoint is fully faithful. By [HA, 7.1.2.7] and [MNN17, 7.6], the categories of the form $\mathrm{Mod}_R \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L},\omega}_{\mathrm{st}})$ for $R \in \mathrm{CAlg}$ are uniquely categorized by the fact that their subcategories of compact objects are unigenic, from which it follows that that the embedding $\mathrm{Perf} : \mathrm{CAlg} \to \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$ has essential image exactly the full subcategory of unigenic 2-rings. Composing the adjunction above with the equivalence $\mathrm{CAlg} \simeq 2\mathrm{CAlg}^{\mathrm{uni}}$, we obtain an adjunction of the form

$$F : 2\mathrm{CAlg}^{\mathrm{uni}} \rightleftarrows 2\mathrm{CAlg} : G$$

where F is the inclusion. We will deduce a formula for G below.

Lemma 6.3. *The right adjoint G' to the inclusion $2\mathrm{CAlg}^{\mathrm{uni}} \hookrightarrow 2\mathrm{CAlg}$ is given by the functor $\mathcal{K} \mapsto \mathrm{Thick}(\mathbf{1}_{\mathcal{K}})$, and the counit is identified with the canonical inclusion $\mathrm{Thick}(\mathbf{1}_{\mathcal{K}}) \subseteq \mathcal{K}$.*

Proof. Let $\mathcal{K} \in 2\mathrm{CAlg}$ arbitrary. As the left adjoint is an inclusion, [HTT, 5.2.2.7] implies that the counit $\epsilon : G(\mathcal{K}) \to \mathcal{K}$ is a terminal object of $2\mathrm{CAlg}^{\mathrm{uni}}_{-/\mathcal{K}}$. For $\mathbf{1}_{\mathcal{K}} \in \mathcal{K}$, consider the thick subcategory $\mathrm{Thick}(\mathbf{1}_{\mathcal{K}}) \hookrightarrow \mathcal{K}$, and note that this inclusion is a morphism in $2\mathrm{CAlg}$ with unigenic domain. Furthermore, the image of any $F : \mathcal{L} \to \mathcal{K} \in 2\mathrm{CAlg}^{\mathrm{uni}}_{-/\mathcal{K}}$ must be contained in $\mathrm{Thick}(\mathbf{1}_{\mathcal{K}})$, as $F(\mathrm{Thick}(\mathcal{S})) \subseteq \mathrm{Thick}(F(\mathcal{S}))$ for $\mathcal{S} \subseteq \mathcal{L}$ any set of objects. Thus, the counit map admits a lift through $G(\mathcal{K}) \to \mathrm{Thick}(\mathbf{1}_{\mathcal{K}})$. Any such lift is split by lifting the canonical inclusion $\mathrm{Thick}(\mathbf{1}_{\mathcal{K}}) \to \mathcal{K}$ through $\mathrm{Thick}(\mathbf{1}_{\mathcal{K}}) \to G\mathcal{K} \to \mathrm{Thick}(\mathbf{1}_{\mathcal{K}})$, which composes to the identity. As any retract of a terminal object must itself be terminal, we have an equivalence $\mathrm{Thick}(\mathbf{1}_{\mathcal{K}}) \simeq G(\mathcal{K})$ and the counit is identified with the canonical inclusion. □

Definition 6.4. We refer to the right adjoint G' above as the functor of *unitation*.

6.B. **The main result.** The following is Corollary C, which is obtained as an immediate consequence of Theorem 5.3.

Corollary 6.5. *For any $\mathcal{K} \in 2\mathrm{CAlg}_{\mathrm{rig}}$ and qcqs spectral scheme X, there is an identification*

$$\mathrm{map}_{2\mathrm{CAlg}_{\mathrm{rig}}}(\mathrm{Perf}_X, \mathcal{K}) \simeq \mathrm{map}_{\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}}((\mathrm{Spec}\,\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\mathcal{K}}}), X)$$

The same holds if X is a Dirac spectral scheme and $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}$ is replaced by $\mathrm{RTop}^{\mathrm{Dir}}_{\mathrm{CAlg}}$.

Proof. By Corollary 2.28 the functor $\mathrm{Spec} : 2\mathrm{CAlg}^{\mathrm{op}}_{\mathrm{rig}} \to \mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg}}$ is fully faithful. The claimed result thus reduces to the identification

$$\mathrm{map}_{\mathrm{RTop}^{\mathrm{loc}}_{2\mathrm{CAlg}}}(\mathrm{Spec}\,\mathcal{K}, \mathrm{Spec}\,\mathrm{Perf}_X) \simeq \mathrm{map}_{\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}}((\mathrm{Spec}\,\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\mathcal{K}}}), X)$$

which is the content of Theorem 5.3. The same argument holds in the Dirac case. □

In many cases of interest, one does not simply have a map from $(\mathrm{Spec}\,\mathcal{K}, \mathcal{R}_{\mathcal{O}})$ to a given spectral or Dirac spectral scheme X, but in fact an identification of this spectrally ringed space as *itself* a spectral or Dirac spectral scheme. We will prove a general "geometrization" result for 2-rings of this form as Theorem 6.6.

Theorem 6.6. *Let $\mathcal{K} \in 2\mathrm{CAlg}_{\mathrm{rig}}$ be a rigid 2-ring and $X \in \mathrm{SpSch}$ admitting an abstract equivalence*

$$\alpha : (\mathrm{Spec}\,\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\mathcal{K}}}) \simeq X \in (\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}})^{[1]}$$

Then there is a fully faithful functor $\mathrm{Perf}_X \to \mathcal{K}$ realizing the equivalence α upon passage to Zariski spectra. The same result holds in the Dirac case.

Proof. The assumption forces X to be qcqs, from which Corollary 6.5 implies that the map α induces a functor $\mathrm{Perf}_X \to \mathcal{K}$. Now let $U \subseteq X$ be any quasi-affine open subset. By the identification of Theorem 2.27, there is an associated commuting square of Karoubi quotients

$$\begin{array}{ccc} \mathrm{Perf}_X & \longrightarrow & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathrm{Perf}_U & \longrightarrow & \mathcal{O}_{\mathcal{K}}(U). \end{array}$$

Moreover, since α is assumed to be an equivalence of spectrally ringed spaces, it induces an identification

$$\Gamma(U, \mathcal{O}_{\mathcal{X}}) \simeq \Gamma(U, \mathcal{R}_{\mathcal{O}_{\mathcal{K}}}) \simeq \mathcal{R}_{\mathcal{O}_{\mathcal{K}}}(\mathcal{U})$$

and hence the associated functor $\mathrm{Perf}_{\Gamma(U,\mathcal{O}_{\mathcal{X}})} \to \mathcal{O}_{\mathcal{K}}(U)$ factors through an equivalence

$$\mathrm{Perf}_{\Gamma(U,\mathcal{O}_{\mathcal{X}})} \simeq \mathrm{Perf}_{\mathcal{R}_{\boldsymbol{O}_{\mathcal{K}}(U)}}$$

with the unitation of $\mathcal{O}_{\mathcal{K}}(U)$. Invoking Lemma 6.3, we learn that the map $\mathrm{Perf}_{\Gamma(U,\mathcal{O}_{\mathcal{X}})} \to \mathcal{O}_{\mathcal{K}}(U)$ must be fully faithful and in fact induces an equivalence between the source and the thick subcategory generated by the unit in the target. Furthermore, since U is assumed to be quasi-affine, the canonical map $\mathrm{Perf}_{\Gamma(U,\mathcal{O}_{\mathcal{X}})} \to \mathrm{Perf}_U$ must be an equivalence. Altogether, we obtain that the composite map

$$\mathrm{Perf}_{\Gamma(U,\mathcal{O}_{\mathcal{X}})} \simeq \mathrm{Perf}_U \to \mathcal{O}_{\mathcal{K}}(U)$$

is fully faithful. Since the quasi-affine open subsets form a basis of X, it follows that the map $\mathrm{Perf}_X \to \mathcal{K}$ may be presented as a limit of fully faithful functors

$$\varprojlim_{U \subseteq X \text{ affine}} [\mathrm{Perf}_U \to \mathcal{O}_{\mathcal{K}}(U)] \in 2\mathrm{CAlg}_{\mathrm{rig}}^{[1]}$$

and is thus itself fully faithful. The argument does not change for the Dirac case.

The statement that the passage to Zariski spectra realizes the map α arises from the fact that the composite

$$(\mathrm{Spec}\,\mathcal{K}, \mathcal{R}_{\mathcal{O}_{\mathcal{K}}}) \to (\mathrm{Spec}\,\mathrm{Perf}_X, \mathcal{R}_{\mathcal{O}_{\mathrm{Spec}\,\mathrm{Perf}_X}}) \to X$$

in $\mathrm{RTop}^{\mathrm{loc}}_{\mathrm{CAlg}}$ realizes the equivalence α, by Theorem 5.3 and the universal property of the relative spectrum. Moreover, the map of spaces $|\mathrm{Spec}\,\mathcal{K}| \to |\mathrm{Spec}\,\mathrm{Perf}_X|$ is a strong spectral quotient map by [San25, Theorem 4.1], implying in particular that it is a homeomorphism if and only if it is injective [San25, Corollary 2.26]. However, since α is an equivalence, it is necessarily injective, yielding the claim. □

We collect some key examples below.

Example 6.7 (Stable module categories). Let G be a finite group of order divisible by p, and k be a field of characteristic p. The *stable module category* of G over k is the object $\mathrm{St}_{kG} \in 2\mathrm{CAlg}_{\mathrm{rig}}$ defined by the Verdier quotient

$$\mathrm{St}_{kG} \simeq \mathrm{Rep}_G(k)/\mathrm{Proj}_G(k)$$

where $\mathrm{Proj}_G(k)$ is the thick subcategory generated by the image of the projective representations in $\mathrm{Rep}_G(k)$ (here by Rep, we really mean the derived category of finite-dimensional representations of G over k). The Balmer spectrum of St_{kG} is computed in [BCR97], where it is shown to be homeomorphic to $\mathrm{Proj}\, H^*(G,k)$. [Bal10b, Proposition 8.8] demonstrates more, namely this homeomorphism is realized on underlying topoi via the comparison map

$$(\mathrm{Spec}\,\mathrm{St}_{kG}, \mathcal{R}_{\mathcal{O}}) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}}\, \mathcal{R}_{\mathrm{St}_{kG}} \in \mathrm{RTop}^{\mathrm{Dir}}_{\mathrm{CAlg}}$$

and the same argument easily shows that the above is an equivalence in locally spectrally ringed spaces. However, as a spectrally ringed space, the object $\mathrm{Spec}^{\mathcal{G}_{\mathrm{Dir}}} \mathcal{R}_{\mathrm{St}_{kG}}$ is in fact an *ordinary* spectral scheme, given by the space $\mathrm{Proj}\ H^*(G,k)$ equipped with the sheaf of spectra

$$\{U(f) \subseteq \mathrm{Proj}\ H^*(G,k)\} \mapsto \mathcal{R}_{\mathrm{St}_{kG}}[f^{-1}].$$

We refer to this spectral scheme as the *spectral support variety* of G over k, and use the notation $\mathcal{V}_G$ to denote it. By the result of Theorem 6.6, there is a fully faithful functor $\mathrm{Perf}_{\mathcal{V}_G} \to \mathrm{St}_{kG}$. Moreover, it can be shown that this embedding is functorial in the group G.

The primary motivation for treating the Dirac spectral setting above is the following example, which shows that Dirac spectral schemes ought also abound in nature.

Example 6.8. (Permutation modules) Let G be a group and $R \in \mathrm{CAlg}^{\heartsuit}$ be an ordinary commutative Noetherian ring. A *permutation $R[G]$-module* is an $R[G]$ module of the form $R[X]$ for $X \in \mathrm{Set}^{BG}$ a G-set. Let $\mathrm{perm}(G,R) \subseteq \mathrm{Mod}^{\heartsuit}_{R[G]}$ denote the additive category of finitely generated permutation $R[G]$-modules. The tensor-product of $R[G]$-modules restricts to $\mathrm{perm}(G,k)$, and the latter is thus an additively monoidal category.

Consider the bounded homotopy category of chain complexes in $\mathrm{perm}(G,k)$, denoted $\mathcal{K}(G,R)$. This is a rigid tensor-triangulated category whose Balmer spectra are computed in [BG25] for the case where R is a field, and in [DG25] for the case where R is characteristic p or a p-torsion free Noetherian ring.

For $G = E$ an elementary abelian group, the Balmer spectrum $\mathrm{Spc}\,\mathcal{K}(G,R)$ equipped with the unique structure sheaf satisfying

$$\mathcal{O}_E : \{U(a) \subseteq \mathrm{Spc}\,\mathcal{K}(G,R) \text{ quasicompact open}\} \mapsto \pi_* \underline{\mathrm{End}}_{\mathcal{K}(G,R)/\langle a \rangle}(\mathbf{1})$$

is shown to be a Dirac scheme in [BG25, Corollary 15.4], [DG25, Corollary 10.12]. For any given enhancement of $\mathcal{K}(G,R)$ to the structure of a 2-ring, this result will also demonstrate that the Dirac-locally spectrally ringed ∞-topos

$$\mathcal{V}_G^{\mathrm{perm}} := (\mathrm{Spec}\,\mathcal{K}(G,R), \mathcal{R}_{\mathcal{O}}) \in \mathrm{RTop}^{\mathrm{Dir}}_{\mathrm{CAlg}}$$

is a Dirac spectral scheme, through which Theorem 6.6 will supply a fully faithful embedding $\mathrm{Perf}_{\mathcal{V}_G^{\mathrm{perm}}} \hookrightarrow \mathcal{K}(G,R)$. We hope to return to this observation in future work.

Finally, we remark that the same proof as in Theorem 6.6 actually supplied a slightly stronger statement.

Theorem 6.9. *Let $\mathcal{K} \in 2\mathrm{CAlg}_{\mathrm{rig}}$ be a rigid 2-ring and $X \in \mathrm{SpSch}_{\mathrm{qcqs}}$ admit a map*

$$\alpha : (\mathrm{Spec}\,\mathcal{K}, \mathcal{R}_{\mathcal{O}_K}) \to X \in (\mathrm{RTop}^{loc}_{\mathrm{CAlg}})^{[1]}$$

such that the associated map $\mathcal{O}_X \to \alpha_* \mathcal{R}_{\mathcal{O}_K}$ is an equivalence in $\mathrm{Shv}(X;\mathrm{CAlg})$. Then there is a fully faithful functor $\mathrm{Perf}_X \to \mathcal{K}$.

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