Derived Reconstruction

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We let $2\mathrm{CAlg} \coloneqq \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})_{\mathrm{rig}}$ the ∞-category of *2-rings* be the underlying $(\infty,1)$-category of *symmetric monoidal, stable, idempotent complete* ∞-categories with a biexact tensor product that are *rigid*, meaning every object is dualizable (here the symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is that of [BGT13, 3.1]). In particular, for any object $\mathcal{K} \in 2\mathrm{CAlg}$, the homotopy category $\mathrm{ho}(\mathcal{K})$ is canonically tensor-triangulated and rigid [Bar+23, 5.12]. To any small tensor-triangulated category $\mathcal{K}_0$, one may equip the basic open sets of its Balmer spectrum $\mathrm{Spc}(\mathcal{K}_0)$ with a "structure presheaf" of triangulated categories [Bal02, §5], given by the following assignment:

$$U(a) \mapsto \mathcal{K}_0(U(a)) \coloneqq (\mathcal{K}_0/a)^\natural$$

where $U(a) \subseteq \mathrm{Spc}(\mathcal{K}_0)$ is the basic open set corresponding to the primes which contain $a \in \mathcal{K}_0$, $\mathcal{K}_0/a$ denotes the Verdier quotient of $\mathcal{K}_0$ by the thick tensor-ideal generated by a, and $(-)^\natural$ denotes idempotent completion.

Our first result demonstrates that for any $\mathcal{K} \in 2\mathrm{CAlg}$, the "structure presheaf" on $\mathrm{Spc}(\mathrm{ho}\,\mathcal{K})$ upgrades to a full structure sheaf valued in 2CAlg, with an appropriate locality condition. We recall the following definition.

Definition 1. [Bal10, 4.1] A tensor-triangulated category $\mathcal{K}_0$ is called *local* if the thick tensor ideal $\{0\} \subseteq \mathcal{K}_0$ is prime.

We now have the following:

Theorem 2. *For $\mathcal{K} \in 2\mathrm{CAlg}$, there is a natural sheaf $\mathcal{O}_\mathcal{K} \in \mathrm{Shv}_{2\mathrm{CAlg}}(\mathrm{Spc}(\mathrm{ho}\,\mathcal{K}))$ such that for any $a \in \mathcal{K}$, $\mathrm{ho}(\mathcal{O}_\mathcal{K}(U(a))) = (\mathrm{ho}\,\mathcal{K}/\langle a\rangle)^\natural$. Furthermore, for every $x \in \mathrm{Spc}(\mathrm{ho}\,\mathcal{K})$, the homotopy category of its stalk $\mathcal{O}_{\mathcal{K},x}$ is a local tt-category.*

Remark 3. Any $x \in \mathrm{Spc}(\mathrm{ho}\,\mathcal{K})$ corresponds to a prime tt-ideal $\mathcal{P} \subseteq \mathcal{K}$, and the homotopy category of the stalk $\mathcal{O}_{\mathcal{K},x}$ is exactly $(\mathrm{ho}\,\mathcal{K}/\mathcal{P})^\natural$.

This motivates the following definition, which we write informally for the purpose of exposition.

Definition 4. The ∞-category $\mathrm{Top}^{\mathrm{loc}}_{2\mathrm{CAlg}}$ of *locally 2-ringed spaces* is the $(\infty,1)$-category whose objects are pairs $(X,\mathcal{O}_X)$ where $X\in\mathrm{Top}$ and $\mathcal{O}_X\in\mathrm{Shv}_{2\mathrm{CAlg}}(X)$ with *local* stalks, with morphisms $f:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ given (roughly) by pairs

$$[f_\# : X\to Y]\in\mathrm{Top}^{\Delta^1},\ [f^\# : \mathcal{O}_Y\to f_*\mathcal{O}_X]\in\mathrm{Shv}_{2\mathrm{CAlg}}(Y)^{\Delta^1}$$

where $f^\#$ induces *conservative* functors on homotopy categories of stalks.

We will henceforth utilise the notation $\mathrm{Spc}(\mathcal{K})$ to refer to the locally 2-ringed space $(\mathrm{Spc}(\mathrm{ho}\,\mathcal{K}),\mathcal{O}_{\mathcal{K}})$. Our next result provides both functoriality and a universal property for $\mathrm{Spc}(\mathcal{K})$ among all locally 2-ringed spaces.

Theorem 5. *The assignment $\mathcal{K}\mapsto\mathrm{Spc}(\mathcal{K})$ promotes to a fully faithful functor $\mathrm{Spc}(-):2\mathrm{CAlg}^{op}\to\mathrm{Top}^{\mathrm{loc}}_{2\mathrm{CAlg}}$. Furthermore, for any $\mathcal{X}\in\mathrm{Top}^{\mathrm{loc}}_{2\mathrm{CAlg}}$, one has the following equivalence*

$$\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{2\mathrm{CAlg}}}(\mathcal{X},\mathrm{Spc}(\mathcal{K}))\simeq\mathrm{Map}_{2\mathrm{CAlg}}(\mathcal{K},\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}))$$

via the map that takes a morphism $f:\mathcal{X}\to\mathrm{Spc}(\mathcal{K})$ to the induced map on global sections of structure sheaves.

We note that the results above have been obtained independently by joint work of Ko Aoki, Tobias Barthel, Tomer Schlank, and Greg Stevenson, using a slightly different formulation.

We go on to apply the machinery above in proving a derived-geometric extension of a classical result of Balmer-Thomason on the reconstruction of coherent schemes from their categories of perfect complexes (stated in full generality as [KP17, 4.2.5]). To formulate the same, we need the following definition.

Definition 6. We write $\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K})\in\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}$ to denote the locally spectrally ringed space given by $(\mathrm{Spc}(\mathcal{K}),\mathbf{End}_{\mathbb{1}}(\mathcal{O}_{\mathcal{K}}))$, where $\mathbf{End}_{\mathbb{1}}:2\mathrm{CAlg}\to\mathrm{CAlg}$ denotes the functor sending a 2-ring to the *endomorphism ring spectrum* of its unit object.

Remark 7. The fact that the spectrally ringed space above is *locally* spectrally ringed is an observation originally made in [Bal10, 6.6]. Furthermore, the proposition preceding this observation implies that one has a functor $\mathbf{LRS}:\mathrm{Top}^{\mathrm{loc}}_{2\mathrm{CAlg}}\to\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}$ by sending $(X,\mathcal{O}_X)\mapsto(X,\mathbf{End}_{\mathbb{1}}(\mathcal{O}_{\mathcal{X}}))$.

Our main result is the following upgraded reconstruction result for a certain class of spectral schemes.

Theorem 8. *Let $\mathcal{X}\in\mathrm{SpSch}^{\mathrm{nc}}$ be a nonconnective spectral scheme whose underlying classical scheme $\mathcal{X}^{\heartsuit}$ is coherent and has affine diagonal. Then there is a canonical map of locally spectrally ringed spaces $\gamma_{\mathcal{X}}:\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_{\mathcal{X}})\to\mathcal{X}$, satisfying the following:*

1. *Any open immersion of an affine spectral subscheme $\iota:\mathrm{Spec}(R)\hookrightarrow\mathcal{X}$, induces an open inclusion $U:=\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R)\hookrightarrow\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_{\mathcal{X}})$, and the restriction $\gamma_{\mathcal{X}}|_U:\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R)\to\mathcal{X}$ is given by a composition $\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R)\xrightarrow{\rho_R}\mathrm{Spec}(R)\xrightarrow{\iota}\mathcal{X}$.*

2. *The map ρ_R is the affinization map associated to the locally spectrally ringed space* $\mathrm{Spc}^{\mathrm{LRS}}(\mathrm{Perf}_R)$*, and in particular on underlying classically ringed spaces it recovers the comparison map* $\mathrm{Spc}(\mathrm{hoPerf}_R) \to \mathrm{Spec}(\pi_0 R)$ *of* [Bal10].

3. *One has a natural equivalence*

$$\mathrm{Map}_{\mathrm{Top}^{\mathrm{loc}}_{\mathrm{CAlg}}}(\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K}), \mathcal{X}) \simeq \mathrm{Map}_{2\mathrm{CAlg}}(\mathrm{Perf}_{\mathcal{X}}, \mathcal{K})$$

for any $\mathcal{K} \in 2\mathrm{CAlg}$.

We remark that parts (1) and (2) of the above theorem demonstrate that the Balmer spectra of spectral schemes is fully determined by the comparison maps on each affine patch, indicating that their Balmer spectra are governed by a "geometric direction" corresponding to the underlying classical scheme of $\mathcal{X}$, and a "homotopy theoretic" direction governed by the affine schemes in any chart.

Remark 9. We indicate a few example applications below.

1. From (1) and (2), one can immediately deduce that for any *connective* spectral scheme satisfying the conditions of the theorem, the comparison map γ of the theorem is surjective using the results of [Bal10, §7].

2. The full reconstruction for classical schemes as stated in [KP17, 4.2.5] is a direct consequence of the reconstruction theorem above, combined with [Bal10, 8.1].

3. Given any locally even periodic spectral schemes whose underlying classical scheme is regular noetherian and satisfies the conditions of the theorem, γ is an equivalence (for example, by reformulating the results of [Mat15, §2] in terms of the comparison maps ρ_R of (1) and (2).

We end by indicating an application of part (3) of the theorem above.

Corollary 10. *Given any locally monogenic 2-ring* $\mathcal{K}$ *such that* $\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K})$ *is itself a coherent nonconnective spectral scheme whose underlying classical scheme has affine diagonal, one has an equivalence of 2-rings*

$$\mathcal{K} \simeq \mathrm{Perf}_{\mathrm{Spc}^{\mathrm{LRS}}(\mathcal{K})}$$

In particular, this yields an equivalence of tensor-triangulated categories upon passage to homotopy categories.

A key example of categories satisfying the above include the principal blocks of any ∞-categorical enhancement of the stable module categories of a finite flat group scheme G over a field k of characteristic $p > 0$, by results of [FP07]. Equivalences of this form enable the computation of invariants in these categories using descent-theoretic techniques based on the associated spectral scheme: these have been utilised in chromatic homotopy theory to great effect, for example in the computation of the Picard group of TMF via an étale descent spectral sequence as in [MS16], or in the classification of certain Azumaya algebras for TMF as in [BMS22]. We end our discussion with two questions:

Question 11. Can the Picard and Brauer groups of the principal blocks for stable module categories be completely computed by a descent spectral sequence based on their associated spectral schemes?

Question 12. Do the spectral schemes appearing in the equivalences above admit natural spectral moduli-theoretic interpretations? I am presently able to provide an affirmative answer only for elementary abelian groups.

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